Math 131 notes

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Contents

Notes also available as PDF.

1 Language of set theory

- We will cover just enough set theory to use later.

- Cardinalities are important for probability. We don't have time to cover probability sufficiently well, so we will not explore the sizes of sets deeply.
- This is known as naïve set theory. We do not define absolutely everything, nor do we push set theory's logical limits. Much.

Goals:

- Impart some of the language necessary for later chapters.
- Practice reasoning in a formal setting.
  - One key aspect is what to do in extreme cases like empty sets.
- Set up straight-forward examples for logic.

2 Basic definitions

To start, we require unambiguous definitions of terms and items. When a term or item is unambiguously defined, it is called *well-defined*.

set An *unordered* collection of *unique* elements.

- Curly braces: $\{A, B, C\}$ is a set of three elements, A, B, and C.
- Order does not matter: $\{\text{cat}, \text{dog}\}$ is the same set as $\{\text{dog}, \text{cat}\}$.
- Repeated elements do not matter: $\{1, 1, 1\}$ is the same set as $\{1\}$.
- Can be *implicit*: $\{x \mid x \text{ is an integer}, x > 0, x < 3\}$ is the same set as $\{1, 2\}$.
- Read the implicit form as "the set of elements x *such that* x is an integer, $x > 0$, and $x < 3$". Or "the set of elements x *where* …"
- Other symbols that sometimes stand for "such that": :, $\ni$ (reversed $\in$)
- Implicit (or set-builder) form can include formula or other bits left of the bar. $\{3x \mid x \text{ is a positive integer}\}$ is the set $\{3, 6, 9, \ldots\}$.

element Any item in a set, even other sets. (Also entry, member, item, *etc.*)

- This is not ambiguous. If something is in a set, it is an item of that set. It doesn't matter if the item is a number or a grape.
- $\{A, \{B, C\}\}$ is a set of *two* elements, A and $\{B, C\}$.
- None of the following are the same: $\{A, \{B, C\}\}$, $\{A, B, C\}$, $\{\{A, B\}, C\}$.

empty set Or null set. Denoted by $\emptyset$ rather than $\{\}$.

- This is a *set* on its own.

- $\{\emptyset\}$ is the set of the empty set, which is not empty.
- Think of sets as bags. An empty bag still is a bag, and if a bag contains an empty bag, the outer bag is not empty.
- Implicit definitions can hide empty sets.
- For example, the set $\{x \,|\, x \text{ is an odd integer divisible by } 2\}$ is $\emptyset$.

singleton A set with only one element.

- $\{1\}$ and $\{\emptyset\}$ both are singletons (or sometimes singleton sets).

3 Translating sets into (and from) English

From English:

- The days of the week:
  - $\{\text{Monday}, \text{Tuesday}, \text{Wednesday}, \text{Thursday}, \text{Friday}, \text{Saturday}, \text{Sunday}\}$
  - Of course, we're using a *representation* of the days and not the days themselves. That is how we reason about things; we model them and represent them by symbols.
- The days when homework is due:
  - $\{25^{\text{th}} \text{ of August}, 1^{\text{st}} \text{ of September}, \ldots\}$
  - We *could* list them all.
  - { every Monday after the 18^{th} of August 2008 until after the 1^{st} of December }
  - Or: $\{x \,|\, x$ is a Monday, x is after the 18^{th} of August, and x is on or before the 1^{st} of December $\}$

To English:

- $\{2, 3, 4\}$:
  - The set containing two, three, and four.
- $\{x \,|\, x \text{ is an integer and } x > 0\}$:
  - The positive integers, also called the counting numbers or the natural numbers.
  - Often written as $\mathbb{J}^+$. The integers often are written as $\mathbb{J}$ (because the "I" form can be difficult to read), rationals as $\mathbb{Q}$ (for quotients), the reals as $\mathbb{R}$.
- $\{2x - 1 \,|\, x \in \mathbb{J}^+\}$

– The set whose members have the form $2x - 1$ where x is a positive integer.
– Cannot list all the entries; this is an *infinite* set.
– Here, the odd integers.

4 Relations

element of The expression $x \in A$ states that x is an element of A. If $x \notin A$, then x is *not* an element of A.

- $4 \in \{2, 4, 6\}$, and $4 \notin \{x \,|\, x \text{ is an odd integer }\}$.
- There is no x such that $x \in \emptyset$, so $\{x \,|\, x \in \emptyset\}$ is a long way of writing $\emptyset$.

subset If all entries of set A also are in set B, A is a subset of B.

superset The reverse of subset. If all entries of set B also are in set A, then A is a superset of B.

proper subset If all entries of set A also are in set B, but some entries of B are *not* in A, then A is a *proper* subset of B.

- $\{2, 3\}$ is a proper subset of $\{1, 2, 3, 4\}$.

equality Set A equals set B when A is a subset of B and B is a subset of A.

- Order does not matter. $\{1, 2, 3\} = \{3, 2, 1\}$.

The symbols for these relations are subject to a little disagreement.

- Many basic textbooks write the subset relation as $\subseteq$, so $A \subseteq B$ when A is a subset of B. The same textbooks reserve $\subset$ for the *proper* subset. Supersets are $\supset$.
- This keeps a superficial similarity to the numerical relations $\leq$ and $<$. In the former the compared quantities may be equal, while in the latter they must be different.
- Most mathematicians now use $\subset$ for any subset. If a property requires a "proper subset", it often is worth noting specifically. And the only non-"proper subset" of a set is the set itself.
- Extra relations are given for emphasis, *e.g.* $\subsetneq$ or $\subsetneqq$ for proper subsets and $\subseteq$ or $\subseteqq$ to emphasize the possibility of equality.
- Often a proper subset is written out: $A \subset B$ and $A \neq B$.
- **I'll never remember to stick with the textbook's notation. My use of $\subset$ is for subsets and not proper subsets.**

5 Translating relations into (and from) English

From English:

- The train has a caboose.
  - It's reasonable to think of a train as a set of cars (they can be reordered).
  - The cars are the members.
  - Hence, caboose $\in$ train
- The VI volleyball team consists of VI students.
  - VI volleyball team $\subset$ VI students
- There are no pink elephants.
  - pink elephants $= \emptyset$

To English:

- $x \in$ today's homework set.
  - x is a problem in today's homework set.
- Today's homework $\subset$ this week's homework.
  - Today's homework is a subset of this week's homework.

6 Consequences of the set relation definitions

Every set is a subset of itself. Expected.

If $A = B$, then every member of A is a member of B, and every member of B is a member of A. This is what we expect from equality, but we did not define set equality this way. Follow the rules:

- $A = B$ imples $A \subset B$ and $B \subset A$.
- Because $A \subset B$, every member of A is a member of B.
- Because $B \subset A$, every member of B is a member of A.

The empty set $\emptyset$ is a subset of all sets. Unexpected! This is a case of carrying the formal logic to its only consistent end.

- For some set A, $\emptyset \subset A$ if every member of $\emptyset$ is in A.
- But $\emptyset$ has no members.
- Thus all of $\emptyset$'s members also are in A.
- This is called a *vacuous* truth.

The alternatives would not be consistent, but proving that requires more machinery that we need.

7 Visualizing two or three sets: Venn diagrams

Also known as *Venn diagrams*.

yes, at some point I will draw some and stick them in the notes.

8 Operations

union The *union* of two sets A and B, denoted by $A \cup B$, is the set consisting of all elements from A and B.

- $A \cup B = \{x \,|\, x \in A \text{ or } x \in B\}$.
- Remember repeated elements do not matter: $\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$.

intersection The *intersection* of two sets A and B, denoted $A \cap B$, is the set consisting of all elements that are in *both* A and B.

- $A \cap B = \{x \,|\, x \in A \text{ and } x \in B\}$.
- $\{1, 2\} \cap \{2, 3\} = \{2\}$.
- $\{1, 2\} \cap \{3, 4\} = \{\} = \emptyset$.

set difference The *set difference* of two sets A and B, written $A \setminus B$, is the set of entries of A that are not entries of B.

- $A \setminus B = \{x \,|\, x \in A \text{ and } x \notin B\}$.
- Sometimes written as $A - B$, but that often becomes confusing.

If A and B share no entries, they are called *disjoint*. One surprising consequence is that every set A has a subset disjoint to the set A itself.

- No sets (not even $\emptyset$) can share elements with $\emptyset$ because $\emptyset$ has no elements.
- So all sets are disjoint with $\emptyset$.
- The empty set $\emptyset$ is a subset of all sets.
- So all sets are disjoint with at least one of their subsets!

Can any other subset be disjoint with its superset? *No.*

8.1 Similarities to arithmetic

Properties of arithmetic:

commutative $a + b = b + a$, $a \cdot b = b \cdot a$

associative $a + (b + c) = (a + b) + c$, $a(bc) = (ab)c$

distributive $a(b + c) = ab + ac$

Which of these apply to set operations *union* and *intersection*? (Informally. Formally we must rely on the properties of *and* and *or*.)

If $C = A \cup B$, then $C = \{x \mid x \in A \text{ or } x \in B\}$. Reversing the sets does not matter, so $C = A \cup B = B \cup A$. The union is **commutative**. Similary, if $D = A \cup (B \cup C)$, we can write D in an implicit form and see that $D = (A \cup B) \cup C$ to see that the union is **associative**.

The same arguments show that set intersection is **commutative** and **associative**.

For the distributive property, which is similar to addition and which to multiplication? A gut feeling is that unions *add*, so try it.

$$\begin{aligned} A \cap (B \cup C) &= \{x \mid x \in A \text{ and } x \in B \cup C\} \\ &= \{x \mid x \in A \text{ and } (x \in B \text{ or } x \in C)\} \\ &= \{x \mid (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)\} \\ &= (A \cap B) \cup (A \cap C) \end{aligned}$$

But with sets, *both* operations distribute:

$$\begin{aligned} A \cup (B \cap C) &= \{x \mid x \in A \text{ or } x \in B \cap C\} \\ &= \{x \mid x \in A \text{ or } (x \in B \text{ and } x \in C)\} \\ &= \{x \mid (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)\} \\ &= (A \cup B) \cap (A \cup C) \end{aligned}$$

The rules of set theory are intimately tied to logic. Logical operations dictate how set operations behave. We will cover the properties of logic in the next chapter.

9 Translating operations into English

To English:

- $(A \cup B) \cap C$

    - The set consisting of members that are in C and either of A or B.
- $(A \cap B) \cup C$
    - The set consisting of members that are in C or in both of A or B.

10 Special operations

The complement and cross-product operations require extra definitions.

10.1 Universes and complements

universe A master set containing all the other sets in the current context.

complement The *complement* of a set A is the set of all elements in a specified universal set U that are *not* in A.

- $A^c = \{x \,|\, x \notin A \text{ and } x \in U\} = U - A$.
- Sometimes written as A' or $\bar{A}$.

- It's not always necessary to define a universal set.
- And there is no "universal" universal set.
- Because $A^c = U \setminus A$, many people avoid the complement completely.
- The complement is useful to avoid writing many repeated $U \setminus A$ operations that share the same universal set.

10.2 Tuples and cross products

tuple An *ordered* collection of elements, (A, B, C).

- When only two elements, this is an *ordered pair*.
- Think of coordinates in a graph, (x, y).
- So $(x, y) \neq (y, x)$ in general (*i.e.* when $x \neq y$).

cross product A *set* of all *ordered pairs* whose entries are drawn from two sets.

- $A \times B = \{(x, y) \,|\, x \in A, y \in B\}$.

Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$.

Then $A \times B = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)\}$ and
$B \times A = \{(b_1, a_1), (b_1, a_2), (b_2, a_1), (b_2, a_2)\}$. Because $(a_1, b_1) \neq (b_1, a_1)$ in general, $A \times B \neq B \times A$ in general.

When does $A \times B = B \times A$?

11 Cardinality and the power set

cardinality The *cardinality* of a set A is the number of elements in A. Often written as $|A|$. The text uses $n(A)$.

- If $A = \{1, 2, 3\}$, then $|A| = 3$.
- What is $|\emptyset|$? 0.

power set The *power set* of a set A is the set of all subsets of A.

- Often denoted as $\mathcal{P}(A)$, but this is used rarely enough that the notation always needs defined.

What is the cardinality of the power set of A?

- What is cardinality of the power set of $\emptyset$?
  - All sets are subsets of themselves, and the empty set is a subset of itself.
  - Then $\mathcal{P}(\emptyset) = \{\emptyset\}$, and $|\mathcal{P}| = |\{\emptyset\}| = 1$.
- What is the powerset of a set with one element, let's say $\{1\}$?
  - There are two subsets, $\emptyset$ and the set itself $\{1\}$.
  - $\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$, and $|\mathcal{P}(\{1\})| = 2$.
- Two elements, say $\{1, 2\}$?
  - $\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.
  - $|\mathcal{P}(\{1, 2\})| = 4$.
- So the powerset with zero entries has size 1, one entry has size 2, two has size 4, ...

What is the cardinality of $A \cup B$?

- Sets do not contain repeated members, so the union cannot be simply the sum of its arguments.
- The intersection contains one copy of all the shared members.
- So to count every item *once* the cardinality of the union is the sum of the cardinalities of the sets minus the cardinality of the intersection.
- $|A \cup B| = |A| + |B| - |A \cap B|$.
- Known as the *inclusion-exclusion principle.*
- Extends to more sets, but you must be careful about counting entries once!

- $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$.

12 Homework

Practice is absolutely critical in this class.

Groups are fine, turn in your own work. Homework is due in or before class on Mondays.

Most of these problems are purely mechanical. This is less work than it appears.

- Section 2.1:
  - Problems 1-8
  - Problems 11 and 17
  - Problems 30 and 32
  - Problems 62, 63, and 66
  - Problems 68, 71, 74, and 78
  - Problem 92
- Section 2.2:
  - Problems 8, 10, 12, 14
  - Even problems 24-34, using the *text*'s definitions of subset and proper subset
- Section 2.3:
  - Problems 1-6
  - Problems 10, 17, 18, 23, 24
  - Problem 31
  - Problem 33, rephrase using complements with respect to the common "universal" set $A \cup B \cup C$.
  - Problems 61, 62
  - Problems 72, 73
  - Problems 117, 118, 121-124

Note that you *may* email homework. However, I don't use Microsoft^{TM} products (*e.g.* Word), and software packages are notoriously finicky about translating mathematics.

If you're typing it (which I advise just for practice in whatever tools you use), you likely want to turn in a printout. If you do want to email your submission, please produce a PDF or PostScript document.