Math 131 notes

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Contents

Notes also available as PDF. ***For now, graphs are only in the PDF version.***

1 Covered So Far

- Problem solving techniques
  - Pólya's principles:
    1. *Understand the problem*
    2. Make a plan
    3. Carry out the plan

4. Look back at what you've done

  - We will be covering algebra and graphs as means for rephrasing and understanding problems.

- Technical vocabulary for mathematics: sets and logic
  - We need a basic vocabulary for describing mathematical entities.
  - Will be using sets to describe equation solutions and functions.
  - Logic underpins everything.
- Number sense and operations from number theory and rational arithmetic
  - Gain some feeling for when numbers and solutions make sense.
  - Is an even number possible? Negative number? *etc.*

2 What Will Be Covered

Topic: Algebra and graphs for mathematical modeling.

A **mathematical model** is a mechanism used for predicting responses from data.

- A climate modefl is a simulation that takes data (mostly satellite sensor readings) and generates concrete predictions.
- A population model often is a formula that takes a limited set of data (*e.g.* initial sizes) and produces a rough estimate of how the populations grow and shrink.

Consider these real life "word problems." The models often can be expressed algebraically; we will be covering a few of these forms.

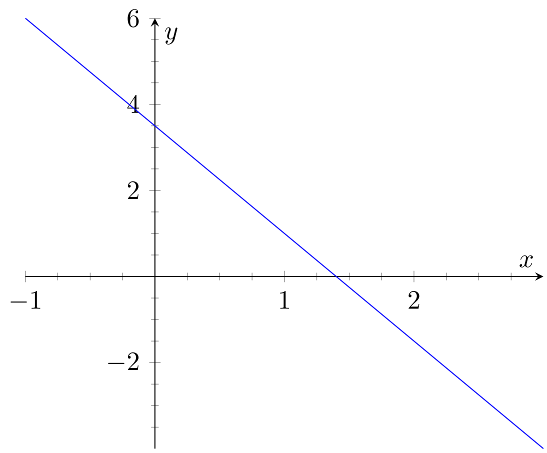
Sometimes an **algebraic** view, working with symbols, is the most useful, and sometimes a **graphical** view, working with plots, is the most useful. Different people and different problems may require different views for fully understanding them. Regardless, each view can serve as a good check.

The algebraic models and relationships we will cover:

- Linear equations
  - Equations in one variable: Useful for simple modeling and for describing algebraic rules. Examples:
    * $x = 5$
    * $65x + 39 = 364$

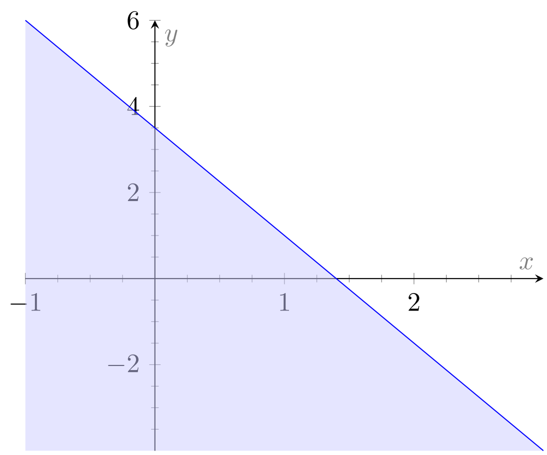
– Equations in multiple variables: Lines, planes, hyperplanes. Examples:

  * $5x + 2y = 7$



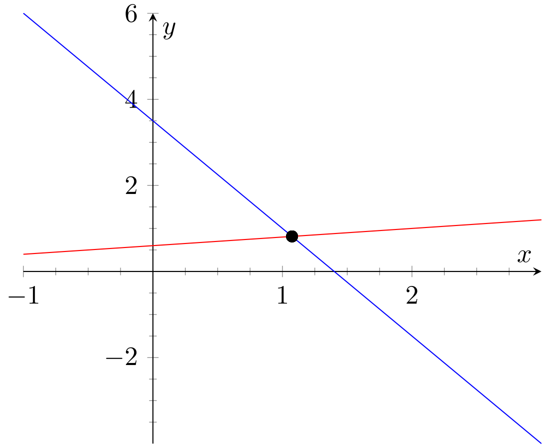
– Inequalities: Constraints on values. Examples:

  * $5x \leq 7$
  * $5x + 2y \leq 7$

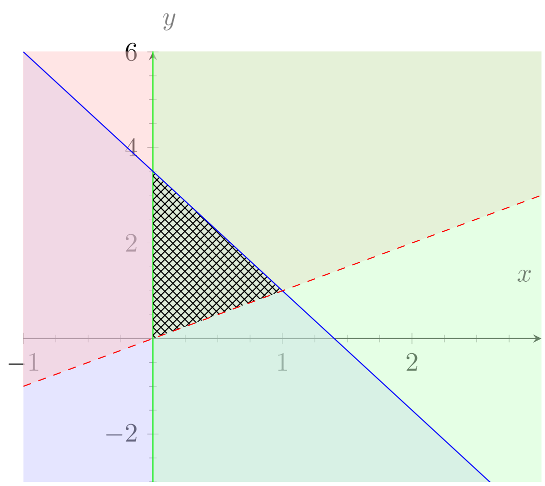


– Systems: Multiple equations or inequalities. Examples:

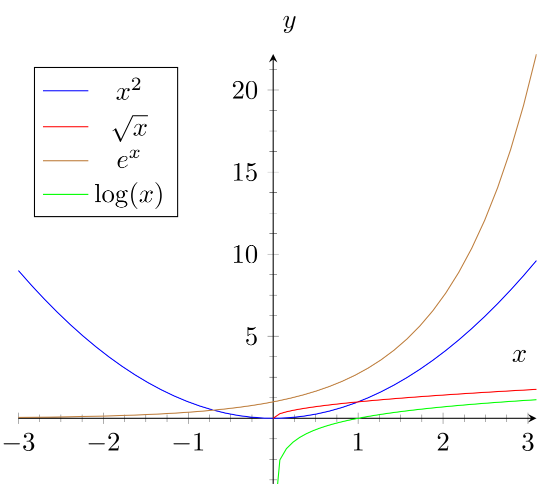
  * $5x + 2y = 7$, $-x + 5y = 3$



$*$ $5x + 2y \leq 7$, $x \geq 0$, $y > x$



- Non-linear equations
  - In one variable: powers, roots, and logarithms



3 An Algebraic Example

Consider the following problem:

> Some proposition loses by 4 votes out of 100 votes cast. How many voted *yes* and how many voted *no*?

One of the first questions to answer is if there is enough information to find a solution. The answer here is *yes*, although it may not be obvious from the description above.

For now, we approach this problem **algebraically** using symbols. This is the approach you remember for general word problems. I won't explain everything here; this is an example we can use throughout. Afterwards, I'll describe a **graphical** approach we can use to quickly see if there is any hope of solution.

The first step in an algebraic approach is to assign **variables** to the unknowns. Here, let Y be the number of *yes* votes, and let N be the number of *no* votes. For now, we do not worry about requiring these to be integers or non-negative numbers. One of the techniques in modeling and algebra is knowing what to ignore and when to ignore it. Often, you ignore some properties of the data. Once you have a final result or solution, you can re-apply those properties to see if it makes sense.

Following Pólya's principles, we next *rephrase the problem* by relating the variables with **equations**.

$$\begin{aligned} \text{loses by 4 votes:} \quad & N = Y + 4 \\ \text{total of 100 votes:} \quad & Y + N = 100 \end{aligned}$$

The general algebraic method for solving systems of equations is to **simplify** them into forms that lead to a result. This is a general *plan* for solving algebraic problems, but it needs broken into sub-plans that may not be obvious when you start.

Here, we can reduce the problem over two variables, Y and N, into a problem over one variable, N, by **substituting** the first equation into the second's left-hand side:

$$\begin{aligned} Y + N &= Y + (Y + 4) && \text{by substitution} \\ &= (Y + Y) + 4 && \text{by the associative property} \\ &= 2Y + 4 && \text{by evaluating } Y + Y. \end{aligned}$$

Now we substitute $2Y + 4$ for $Y + N$ and transform both sides of $2Y + 4 = 100$:

$$\begin{aligned} 2Y + 4 &= 100, && \text{inital equations} \\ (2Y + 4) - 4 &= 100 - 4, && \text{subtract the same from equal quantities} \\ 2Y + (4 - 4) &= 96, && \text{associative property} \\ 2Y + 0 &= 96, && \text{additive inverse} \\ 2Y &= 96, && \text{additive identity} \\ \frac{1}{2} \cdot 2Y &= \frac{1}{2} \cdot 96, && \text{mult. equal quantities by the same} \\ 1 \cdot Y &= 48, && \text{mult. inverse and evaluation} \\ Y &= 48, && \text{mult. identity.} \end{aligned}$$

Then $Y = 48$ and $N = 100 - 48 = 52$. So the final path by which we solved the problem:

1. Rephrase the problem algebraically.
2. Substitute to eliminate one variable, simplifying the problem.
3. Solve a linear equation in one variable.
4. Then substitute back to obtain the other variable.

One of the primary topics we will cover is when to skip all the intermediate steps. Many of the algebra rules we and the text present are designed to allow skipping from the two equations directly to $2Y + 4 = 100$ and then more quickly to $Y = 48$. The reasons given above are purely general, and algebraic notation provides the means of applying those reasons generally.

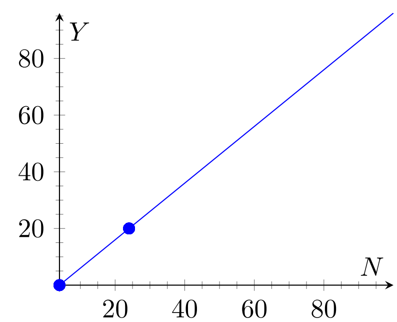
Note that the answer makes sense according to what we know of the problem domain. You cannot have fractional or negative votes. The results are positive integers, so they make sense.

For an example where the number theory we covered helps identify a incorrect data, consider a slight variation where $N = Y + 3$. From the solution above, we know that the difference will satisfy $2Y + 3 = 100$. If Y is an integer, we know that $2 \mid 2Y$ and $2 \mid 100$. But $2 \nmid 3$, so we know this equality cannot have an integer solution. So if our initial method is carried out correctly, *i.e.* we substituted $N = Y + 3$ into $N + Y = 100$ correctly, we know that the problem's initial data *must* be incorrect. This is what I mean by **number sense**.

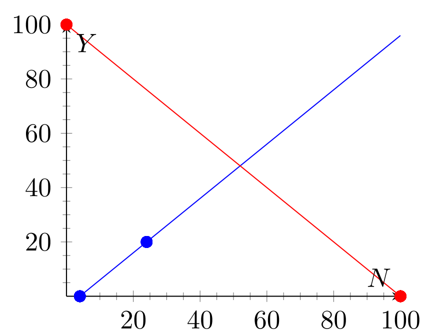
4 The Example's Graphical Side

Thinking of the equations above as relationships, we can plot them on an N-Y graph. I will go into lines and linear forms later. For now, it suffices to remember Euclid's axiom that two points define a line.

For $N = Y + 4$, consider the points where $Y = 0$ and $Y = 20$. These give the values $N = 4$ and $N = 24$. Drawing a line between these points



Now consider $N + Y = 100$. Two points suggest themselves immediately, one at $N = 0$ and one at $Y = 0$. Adding a line through these points:



Even just sketching these without being two precise shows us that the solution may make sense. One line slopes up and one slopes down, so they will intersect somewhere. And a quick sketch shows they intersect with both variables taking positive numbers. This style of **graphical reasoning** often helps show if a solution is possible or impossible. A quick sketch does not show that the variables are positive *integers*, but the sketch does justify carrying out the algebraic work.

5 Definitions

Now for the painful part. We need a common set of definitions.

- An **algebraic expression** is a phrase containing variables, numbers, operations, and groups (parentheses). Examples:
  - $5x$
  - $763 + 873672 + (-77 + 232)$
  - $\sqrt{52x + \sqrt{11 + \sqrt{22y}}}$
- An **equation** relates two algebraic expressions by equality. In some contexts, these also are **equalities** or **identities**. Examples:
  - $5x = 763 + 873672 + (-77 + 232)$
  - $\sqrt{52x + \sqrt{11 + \sqrt{22y}}} = 77z$
- An **inequality** relates two algebraic expressions by a comparison. Examples:
  - $5x < 763 + 873672 + (-77 + 232)$
  - $\sqrt{52x + \sqrt{11 + \sqrt{22y}}} \geq 77z$
- More generally, equations and inequalities are called **relations**. Relations also include negated equalities and inequalities ($x \neq 4$, $36x + 93 \not> 39$).
- A **variable** is a symbol standing for a number or quantity. Variables can be **known** or **unknown**.
  - When solving $5x + 7 = 83$, the variable x is considered an **unknown**.
  - But sometimes repeating a long expression, *e.g.* $\frac{\sqrt{238x+\sqrt{281y}}}{98z}$, becomes cumbersome, and you replace it with with a **known** variable rather than writing it again and again.
- The **degree of a variable** in an expression is the largest exponent applied to the variable. Examples:
  - x has degree one, or is first degree;
  - x^2 has degree two, or is second degree; *etc.*

  The **degree of an expression** is the largest degree of any variable in that expression. Examples:
  - $5x + 8$ has degree one, or is first degree;
  - $x^2 + 5x + 8$ has degree two, or is second degree;
  - $29x^2 + 38219y + 91z^3$ has degree three because of z^3.

There is an unusual corner-case where the context matters. In an expression like x^2y^3, the degree can be two, three, or five depending on other constraints and considerations. If you know nothing more about the context, then often x^2y^3 is considered of third degree because of the y^3 term. We won't worry about these situations.

- The **solution set** is the set of solutions, or the set of variable values that satisfy the given relation (equation, inequality, *etc.*). Examples:
  - $x = 5$ has a solution set of $\{5\}$ for x.
  - $N = Y + 3$, $N + Y = 100$ has the solution set $\{(52, 48)\}$ for the *pair* (N, Y).
- **Equivalent equations** are equations with the same solution sets. Solving equations algebraically consists of transforming equations into simpler, equivalent equations. For example $2Y + 4 = 100$ is equivalent to $2Y = 96$ and $Y = 48$.

  Again, a bit of context can make a difference. For example, $x = 5$ and $y = 5$ have the same solution set, but there are contexts where they are not truely equivalent. We could define equivalence in a way to handle this (*c.f.* β-reduction (beta reduction) in programming language theory), but that's beyond our scope.
- **Equivalent inequalities** and relations are defined similarly.

6 Algebraic Rules for Transformations Between Equivalent Equations

- Adding (or subtracting) an equal quantity to (or from) both sides.
- Multiplying an equal, **non-zero** quantity on both sides.
- Dividing or multiplying by the reciprocal of a **non-zero** quantity on both sides.
- Applying arithmetic properties to rearrange expressions.
- Substitution of like relations.

Each of these keeps the solution set **invariant** or unchanging. Invariance is a *very* powerful property. (See the story of Emmy Noether, who used the properties of invariants to fundamentally change not only abstract algebra but also mathematical physics.)

We won't *prove* these, but we will provide general examples to justify them.

- Adding / subtracting the same quantity:

$$\begin{aligned}
x + 15 &= 20 \\
(x + 15) - 15 &= 20 - 15 && \text{subtracting 15 from both sides} \\
x + (15 - 15) &= 5 && \text{associative property and evaluation} \\
x + 0 &= 5 && \text{additive inverse} \\
x &= 5 && \text{additive identity}
\end{aligned}$$

  Sometimes we may add a *variable* to each side. For example:

$$\begin{aligned}
x - y &= 5 - y \\
(x - y) + y &= (5 - y) + y \\
x &= 5
\end{aligned}$$

  demonstrates that the value of x does not depend on y.

- Multiplying (or dividing by) an equal, known **non-zero** quantity on both sides:

$$\begin{aligned}
5x &= 1 \\
\frac{1}{5} \cdot (5x) &= \frac{1}{5} \cdot 1 \\
(\frac{1}{5} \cdot 5) \cdot x &= \frac{1}{5} \\
1 \cdot x &= \frac{1}{5} \\
x &= \frac{1}{5}
\end{aligned}$$

  In this case, we are working with a constant $5 \neq 0$, so taking the reciprocal (or dividing) is well-defined.

- Multiplying (or dividing by) an equal, **unknown** quantity on both sides. We must take care not to divide by zero. Consider

$$xy = 1.$$

  Here, we **know** that $y \neq 0$ and $x \neq 0$, otherwise $xy = 0$. So here it is safe to move y over as in

$$x = \frac{1}{y}.$$

  But in

$$xy = z$$

  for some **unknown** z, we cannot say if x or y are zero!

- Arithmetic manipulations. Evaluating expressions and applying associative, commutative, and distributive properties to simplify the expression. Commonly referred to as "collecting like terms," or gathering all coefficients of the same variable.

$$
\begin{aligned}
2 + (3 + x) - 7 + 2(x + y) &= -2 + 3x + 2y \\
(x^2 + 11) + x(3x + 4) &= 4x^2 + 4x + 11
\end{aligned}
$$

- Substituting like quantities. We used this in the example above to reduce from a two variable system,

$$
\begin{aligned}
Y + N &= 100, \text{ and} \\
N &= Y + 4,
\end{aligned}
$$

to the single equation $2Y + 4 = 100$.

7 Transformation Examples

See the text's Examples 1, 2, and 3 in Section 7.1. For example 3, you can add the fractions directly without multiplying by the common denominator as well. Multiplying the second fraction by $\frac{3}{3}$ gives

$$\frac{(x + 7) + 3(2x - 8)}{6} = \frac{7x - 17}{6} = -4.$$

8 Manipulating Formulæ by Transformations

Consider an equation for the perimeter of a rectangle, $P = 2L + 2W$. When you need to compute one variable from the others, treat the variables you know as numbers. To compute the length L given the perimeter P and width W, classify

- P and W as *known* variables, and
- L as the *unknown* variable.

To find a formula for L, treat P and W as if they were numbers and solve for L.

$$
\begin{aligned}
&2L + 2W = P, \\
&\qquad 2L = P - 2W && \text{by subtracting } 2W \text{ from both sides, and} \\
&\qquad L = \frac{P - 2W}{2} && \text{by dividing by the non-zero constant two.}
\end{aligned}
$$

9 Homework

Notes also available as PDF.

Practice is absolutely critical in this class.

Groups are fine, turn in your own work. Homework is due in or before class on Mondays.

- Exercises for 7.1:
  - 1, 2, 3, 4
  - 9, 10, 17, 18, 25, 26, 36, 37
  - Assume no variable is zero: 61, 62, 68, 69
  - 76
- Exercises for 7.2:
  - 21, 24, 26
  - 43 (Example 5 is six pages before this problem in my text)
- **Exercises for 8.2: delayed until next week**

Note that you *may* email homework. However, I don't use Microsoft$^{\text{TM}}$ products (*e.g.* Word), and software packages are notoriously finicky about translating mathematics.

If you're typing it (which I advise just for practice in whatever tools you use), you likely want to turn in a printout. If you do want to email your submission, please produce a PDF or PostScript document.