Solutions for the first week's homework Math 131

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Also available as PDF.

1 Notes on received homeworks

- The goal of homeworks is practice on the topics covered in the text and in class. If you're unsure how to tackle one problem, look at the problems nearby or at examples. One may be more clear to you and help you with the assigned problem.
- I have office hours now. Monday and Wednesday 1.30pm to 2.30pm (or possibly later) in the Math Lab down the hall.
- Be sure to read the **entire** problem. Many submissions contained only partial answers even when it was clear you understood the mechanism.
- With problems involving large numbers, expect most calculators and computer software to break. Try to *check* results using properties of the input numbers. For example the product of two numbers with units digit 1 also has units digit 1. Or that the product of two d digit numbers has either 2d or 2d 1 digits. (Think about long-hand multiplication to find these and other properties.)
- If there are questions about which problems were assigned or what the problem is asking, contact me even if it's the night before the homework is due! I may not respond instantly, but it's worth a shot.
- Because there was apparent confusion over which problems were assigned, I will start providing the homework on a separate page as well as directly in the notes.
- In general, writing out steps cushions the blow if the result is incorrect. And writing out *reasons* helps even more. If your homework must be late, reasoning in your own style and words shows you did not just copy solutions. This class is as much about the method of thinking and communicating as it is about the final results!

• Remember that homework is one 20% chunk. But there will be 14 or 15 assignments. Each is at *most*... And if there are 10-20 problems per assignment, then each assignment is at most... This is another reason why homeworks are frequent. The impact of each assignment is a little less when there are many.

2 Exercises for Section 1.1

2.1 Even problems, 2-12

- 2 Deductive. The "if-then" rule about medicine is a premise that is immediately applied.
- 4 Inductive. The three children are examples, but there is no rule dictating birth gender.
- 6 Deductive. An "if-then" rule is given and applied.
- 8 Deductive. The rule is implicit, but the conclusion is derived from data and rules rather than repeated examples.
- 10 Inductive. Only repeated observations are used to justify the conclusion.
- 12 Inductive. Again, only observations enter into the reasoning.

3 Explain the "trick" of Section 1.1's example

A list of numbers as in Section 1.1 (2, 9, 16, 23, 30) does not mean anything on its own. The context before this example implies that one should look for an arithmetic relationship.

The "trick" is that a premise is withheld. As in poorly written mystery novels, crucial information is not available.

All reasoning is based on premises (hypotheses, suppositions, *etc.*) wether implicit or explicit. "Trick" questions like Section 1.1's example rely on misleading you into using an incorrect implicit premise.

4 Exercises for Section 1.2

4.1 Problems 2, 9, and 10

4.1.1 Problem 2

| 3 | | |
|-----|-----------|---|
| 14 | 11 | |
| 31 | 17 | 6 |
| 54 | 23 | 6 |
| 83 | 29 | 6 |
| 118 | 35 | 6 |
| 159 | 41 | 6 |

4.1.2 Problem 9

The formula provided in the text is of order 4, or in other words the highest power of the argument n is n^4 . (Another phrase for this is that the formula is *quartic.*) We expect to need 4 columns to the right of the original sequence (1, 2, 4, 8, 16, 31) to reach an arithmetic sequence.

| points | regions | $\Delta^{(1)}$ | $\Delta^{(2)}$ | $\Delta^{(3)}$ | $\Delta^{(4)}$ |
|--------|---------|----------------|----------------|----------------|----------------|
| 1 | 1 | | | | |
| 2 | 2 | 1 | | | |
| 3 | 4 | 2 | 1 | | |
| 4 | 8 | 4 | 2 | 1 | |
| 5 | 16 | 8 | 4 | 2 | 1 |
| 6 | 31 | 15 | 7 | 3 | 1 |
| 7 | 57 | 26 | 11 | 4 | 1 |
| 8 | 99 | 42 | 16 | 5 | 1 |

The formula provided is

$$R(n) = \frac{1}{24} \left(n^4 - 6n^3 + 23n^2 - 18n + 24 \right).$$

One can compute this directly with any method to verify that the answer is 99.

One convenient way to rewrite a polynomial for evaluation is *Horner's rule*. Horner's rule applies the distributive property of multiplication over addition to pull factors of n out of subexpressions. This rule not only is faster when using a calculator, it also incurs fewer rounding errors when n is not an integer. Applying Horner's rule,

$$R(n) = \frac{1}{24}((n^3 - 6n^2 + 23n - 18)n + 24)$$

$$R(n) = \frac{1}{24}(((n^2 - 6n + 23)n - 18)n + 24)$$

$$R(n) = \frac{1}{24}((((n - 6)n + 23)n - 18)n + 24)$$

Subsituting 8 we find that

$$R(8) = \frac{1}{24}((((8-6) \cdot 8 + 23) \cdot 8 - 18) \cdot 8 + 24))$$

= $\frac{1}{24}(((2 \cdot 8 + 23) \cdot 8 - 18) \cdot 8 + 24))$
= $\frac{1}{24}((39 \cdot 8 - 18) \cdot 8 + 24))$
= $\frac{1}{24}(294 \cdot 8 + 24))$
= $\frac{1}{24}(2376).$

Dividing directly again verifies the result is 99, but a technique to avoid the division is recognizing that 2376 = 2400 - 24. Then

$$R(8) = \frac{1}{24} (2376) = \frac{1}{24} (2400 - 24) = 100 - 1 = 99.$$

4.1.3 Problem 10

The problem is of order 2 (or is quadratic), so we expect two columns beyond the initial sequence.

| n | $n^2 + 3n + 1 = (n+3)n + 1$ | $\Delta^{(1)}$ | $\Delta^{(2)}$ |
|----------|-----------------------------|----------------|----------------|
| 1 | 5 | | |
| 2 | 11 | 6 | |
| 3 | 19 | 8 | 2 |
| 4 | 29 | 10 | 2 |
| 5 | 41 | 12 | 2 |

Substituting 5 into (n+3)n+1 produces $(5+3) \cdot 5 + 1 = 8 \cdot 5 + 1 = 41$, verifying the result.

4.2 Problems 14 and 16

4.2.1 Problem 14

There are two reasonable ways to extend the left pattern. Either is reasonable, and both demonstrate the same property.

The first prepends 10 to each number on the left. The resulting pattern is

 $101 \times 101 = 10\ 201,$ $10\ 101 \times 10\ 101 = 102\ 030\ 201, \text{ and}$ $1\ 010\ 101 \times 1\ 010\ 101 = 1\ 020\ 304\ 030\ 201.$

The second possiblity "reflects" the number across the leading or trailing 1. The resulting pattern is

$$101 \times 101 = 10\,201,$$

$$10\,101 \times 10\,101 = 102\,030\,201, \text{ and}$$

$$101\,010\,101 \times 101\,010\,101 = 10\,203\,040\,504\,030\,201.$$

The common property is that squaring a number with alternating 1 and 0 digits

With these short sequences, the zeros only serve to make the pattern more obvious. Note that $11^2 = 121$ and $111^2 = 12321$. This pattern will break after the central digit is 9. Why?

Note that computing $101\,010\,101^2$ with common desktop computers may produce $10\,203\,040\,504\,030\,200$. The last digit falls off the end of how computers represent floating-point numbers. Computing in integers on "32-bit" computers may produce **28** if the calculation wraps around the 32-bit boundary.

This is one reason why looking for patterns and developing a number sense is important. Errors in calculated results depend on the method used for calculation. Most programs or devices do not explain their methods, so recognizing patterns and other properties are important to prevent being misled.

4.2.2 Problem 16

The next line could well be

 $1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1 = 5^2$.

One method for verifying the result is simple calculation.

Another is to rearrange the problem slightly to show the pattern

$$1 = (\sum_{i=1}^{1} i) + (\sum_{i=1}^{1-1} i) = 1^2,$$

$$1 + 2 + 1 = (\sum_{i=1}^{2} i) + (\sum_{i=1}^{2-1} i) = 2^2,$$

$$1 + 2 + 3 + 2 + 1 = (\sum_{i=1}^{3} i) + (\sum_{i=1}^{3-1} i) = 3^2, \text{ and}$$

$$1 + 2 + 3 + 4 + 3 + 2 + 1 = (\sum_{i=1}^{4} i) + (\sum_{i=1}^{4-1} i) = 4^2.$$

Using the formula $\sum_{i=1}^{n} i = n(n+1)/2$, the n^{th} middle form is

$$\frac{n(n+1)}{2} + \frac{(n-1)n}{2} = \frac{n^2 + n + n^2 - n}{2} = n^2.$$

So the fifth term is indeed 5^2 .

4.3 Problems 29 and 30

4.3.1 Problem 29

There are two clear ways to extend the formula S(n) = n(n+1)/2 into a formula for the sum $2 + 4 + 6 + \cdots + 2n$.

One is to recognize that $2 + 4 + 6 + \dots + 2n = 2(1 + 2 + 3 + \dots + n) = 2S(n) = n(n+1)$. Rephrasing the original problem using summation notation, we have used $\sum_{i=1}^{n} 2i = 2\sum_{i=1}^{n} i = n(n+1)$.

Another is to consider the sum $2+4+6+\dots+2n = (1+1)+(2+2)+(3+3)+\dots+(n+n) = (1+2+3+\dots+n)+(1+2+3+\dots+n) = S(n)+S(n) = 2S(n) = n(n+1).$ In summation notation, $\sum_{i=1}^{n} 2i = \sum_{i=1}^{n} (i+i) = (\sum_{i=1}^{n} i) + (\sum_{i=1}^{n} i) = n(n+1).$

4.3.2 Problem 30

This problem is about phrasing mathematical problems in a way that respects the order of operations.

Some possibilities are the following:

• The square of the sum of the integers from 1 to n equals the sum of the cubes of those same numbers.

- The square of the sum of the first n whole numbers is the sum of their cubes.
- Squaring the sum of the integers from 1 to n is the same as summing their cubes.

The key is that on the left a sum is squared, while on the right cubes are added.

4.4 Problems 32, 39, and 51

This sequence of problems demonstrates similar points in different ways. They are all related to each other and to Problem 16.

4.4.1 Problem 32

Each column (or row) of the blocked triangles represents the integers 1, 2, 3, and 4 by the number of blocks in the column (or row). The total number of blocks in each triangle is 1+2+3+4.

When flipped and combined, the total number blocks is the sum of the two triangles, or $2 \cdot (1+2+3+4)$. The combined figure is a rectangle consisting of $4 \cdot 5$ blocks. So $2(1+2+3+4) = 4 \cdot 5$, or $1+2+3+4 = (4 \cdot 5)/2$.

A better diagram would replace the left-most arrow with an addition operator (+).

4.4.2 Problem 39

Drawing out the dots demonstrates the solution directly.

Or use the result of Problem 16. Note that the first sum, $\sum_{i=1}^{n} i_i$, is the n^{th} triangular number and that the second sum, $\sum_{i=1}^{n-1} i_i$, is the $(n-1)^{\text{th}}$ triangular number. Thus Problem 16 demonstrated that the sum of two consecutive triangular numbers is a square.

4.4.3 Problem 51

Either draw a few consecutive figures from Problem 39 or use Problem 16.

4.5 **Problem 49**

The first pattern to observe is they are all of the form p(n)/2, where p(n) is some polynomial of n. The next pattern is that the numerator $p(n) = n(a \cdot n - b)$ for integers a and b. Then both a and b increase by one when adding a side.

Using these patterns, the n^{th} nonagonal number is

$$N(n) = \frac{n(7n-5)}{2}.$$

Substituting 6, $N(6) = 6(7 \cdot 6 - 5)/2 = 3(42 - 5) = 111$, adding more evidence to the conjecture.

4.6 Problems 51 and 54

4.6.1 Problem 51

Oops. I think meant to give Problem 52 rather than repeat Problem 51, but that's my fault.

4.6.2 Problem 54

Filling in a few values,

| n | T(n-1) | 3T(n-1) + n |
|---|--------|-------------|
| 2 | 1 | 5 |
| 3 | 3 | 12 |
| 4 | 6 | 22 |
| 5 | 10 | 35 |

The first, 5, suggests the five-sided pentagon that produces the second pentagonal number. Later numbers add additional evidence.

One could prove the relationship by expanding 3T(n-1) + n and simplifying the expression, or

$$3T(n-1) + n = 3\left(\frac{(n-1)(n-1+1)}{2}\right) + n$$
$$= \frac{3n^2 - 3n}{2} + n$$
$$= \frac{3n^2 - 3n + 2n}{2}$$
$$= \frac{3n^2 - n}{2}$$
$$= \frac{n(3n-1)}{2}$$
$$= P(n).$$