Discrete Mathematics I (Math 131)
Virginia Intermont College

Jason Riedy

Fall semester, 2008

These pages are available as PDF, either as one growing PDF document for the entirety or as individual documents for each session's notes.

If you have difficulties viewing these or have particular accessibility needs, please mail me at `jason@acm.org`.

Contents

Part I

Introduction

Chapter 1

Syllabus

1.1 Discrete Mathematics I

Initial home page: http://jriedy.users.sonic.net/VI/math131-f08/

Meets MWF 10.00am-10.50am in room 210 on the second floor of the J. F. Hicks Memorial Library.

The original syllabus is available, and notes will be posted as available.

Homework problems are posted in each session's notes.

1.2 Goals

- Gain fluency in and practice with the language that is mathematics.
- Apply mathematical reasoning to historical and modern problems.
- Explore problem solving skills within the realm of discrete mathematics.

1.3 Instructor: Jason Riedy

- email: Jason Riedy <`jason@acm.org`>
- instant messages (sometimes): `jason.riedy@gmail.com`
- office hours: MW 1.30pm-2.30pm in the Math Lab (see Section 47 below) or by appointment. Or you may find me many afternoons at Java J's in Bristol or Zazzy'z in Abingdon.

1.4 Text

Miller, Charles D.; Heeren, Vern E.; Hornsby, John; and Morrow, Margaret L. Mathematical Ideas, tenth edition. Addison Wesley, 2003. ISBN 0-321-16808-9

1.5 Grading

Standard 10-point scale, 3 points on either side for -/+ grades.

The homework is 20%, three mid-term exams are 20% each, and the final counts for 40%. This adds to 120%; the final counts as two 20% scores, and the lowest 20% score is dropped.

1.6 On homework

Some problems will be given in every class. The week's problems will be collected on the following Monday.

Mathematics is a social endeavour. Groups are encouraged, but everyone must turn in their own work. At some point, you will be asked to present a homework problem and its solution to the class.

Also, there may be solutions available for problems. But try tackling the problem **yourself** (or with your group) first. Practice is important.

Write out sentences and not sequences of expressions. Explain your approaches. This class is as much about the reasoning process as the results.

1.7 Submitting homework

Groups are fine, turn in your own work. Homework is due in or before class on Mondays.

Note that you *may* email homework. However, I don't use Microsoft™ products (*e.g.* Word), and software packages are notoriously finicky about translating mathematics.

If you're typing it (which I advise just for practice in whatever tools you use), you likely want to turn in a printout. If you do want to email your submission, please produce a PDF or PostScript document.

Chapter 2

Syllabus schedule

Chapters 1, 2, and 3 Scheduled for 18 August through 17 September. View the abstract sections as a chance to practice problem solving techniques. We will find applications shortly. Roughly two weeks on the first chapter, then one week each on chapters 2 and 3.

First exam Scheduled for 19 September.

Chapters 4, 5, and 6 Scheduled for 22 September through 29 October.

Second exam Scheduled for 31 October.

Chapters 7 and 8 Scheduled for 3 November through 26 November.

Third exam Scheduled for 28 November.

Review Scheduled for 1 and 3 December.

Final exam Official time: Wednesday, 10 December from 10.30am until 12.30pm.

Part II

Notes for chapters 1, 2, and 3

Chapter 3

Notes for 18 August

Notes also available as PDF.

3.1 Syllabus and class mechanics

The original syllabus is available.

3.2 Introductions

- My background
- Majors in class

3.3 Inductive and deductive reasoning

Inductive making an "educated" guess from prior observations.

Deductive if premises are satisfied, conclusion follows.

History:

- Old example: Egyptian papyri (1900bc-1800bc)
- Arithmetic table, list of worked problems, used as a text.
- Solve "new" problems by finding similar ones and imitating them.
- Continued through to Greek times (Euclid's Elements, 300bc)
  - Geometry replaced explicit counting.

- Even then, no algebra and little abstraction.
- Algebra:
  - 500bc for babylonians!
  - 200ad for greeks (Diophantus of Alexandria)
  - Spread widely from Persians, Muhammad ibn Mūsā al-khwārizmī in 820ad.
    - (non-translation of his book's title gave "algebra", his name gives "algorithm")

Mathematics is a combination of both forms of reasoning in no particular order.

Problems to find inductive

Problems to prove deductive

Finding a proof both!

3.4 Inductive

1 + 2 + 3	= 6	= 3 * 4 / 2
1 + 2 + 3 + 4	= 10	= 4 * 5 / 2
... + 5	= 15	= 5 * 6 / 2

So what is the sum of the first 50?

$$50 * 51/2 = 25 * 51 = 25(25 \cdot 1) + 250(50 \cdot 5) + 1000(50 \cdot 20) = 1275$$

Integer sequence superseeker from AT&T gives 250 results matching $(3, 10, 15)$. Some of the sequences are built similarly.

[**NOTE** Text uses "probable". Don't do that. There's no probability distribution defined over the choices, so no one choice is more "probable".]

Only takes a single **counterexample** to ruin a perfectly wrong theory.

Must be very careful and define what we mean and want. These are the hypotheses or premises.

What is the premise above?

Could we use an extreme case to check possibilites? (What is the sum of 1?)

3.5 Deductive

Start with a collection of premises and combine them to reach a result. Note: the rules for combining these also are premises!

Knowing to distinguish a “general principle” from a hypothesis takes time and perspective. That’s part of what we’re covering, but don’t worry much about it now.

Typical patterns:

- if {premise} then {conclusion}
- {premises} therefore... or hence...

Used before algebra (Greek geometry), but algebra really helps.

$$\begin{aligned} S &= 1 + 2 + 3 + \ldots + n \text{ (Note use of ellipsis)} \\ S &= n + (n-1) + (n-2) + \ldots + 1 \text{ (Reversed, the sum is the same)} \\ &\text{(add the two)} \\ 2S &= (n+1) + (n+1) + (n+1) + \ldots + (n+1) = n(n+1) \end{aligned}$$

Chapter 4

Notes for 20 August

Notes also available as PDF.

4.1 Review: Inductive and deductive reasoning

Inductive making an "educated" guess from prior observations.

Deductive if premises are satisfied, conclusion follows.

Mathematics is a combination of both forms of reasoning in no particular order.

- Problems to find: inductive
- Problems to prove: deductive
- Finding a proof... both!

Recall examples:

- Example of inductive reasoning: Extending a sequence from examples.
- Example of deductive reasoning: Deriving a rule for computing a sequence.

Always take care with your premises. Be sure you understand the framework before exploring with guesses.

4.2 Inductive reasoning on sequences

Purpose: Define some terminology. See how different sequences grow.

Sequence list of numbers

Term one of the numbers in a list

Examples:

- 3, 5, 7, 9, 11, ...
- 4, 12, 36, 108, ...

(Elipsis is **three** dots and is **not** followed by a comma. Text's use is incorrect on page 10.)

Two common types of sequences:

Arithmetic Defined by an initial number and a constant increment.

Geometric Defined by an initial number and a constant multiple.

In our examples:

- 3, 5, 7, 9, 11, ... : Arithmetic, starts with 3, incremented by 2.
- 4, 12, 36, 108, ... : Geometric, starts with 4, multiplied by 3.

On growth:

- Note how the arithmetic sequence's growth is "smooth", linear.
- The geometric sequence grows much more quickly, exponential.

4.3 A tool for sequences: successive differences

Technique is useful for finding an arithmetic sequence buried in a more complicated appearing sequence of numbers.

This is an example of reducing to a known, simpler problem. We will explore this and other general problem solving methods shortly.

Simple example with an arithmetic sequence:

3	
5	2
7	2
9	2
11	2

Note that the last column provides the increment.

Another example, not directly arithmetic:

2			
6	4		
22	16	12	
56	34	18	6
114	58	24	6

The third column is an arithmetic sequence.

To obtain the next term, fill in the table from the right:

2			
6	4		
22	16	12	
56	34	18	6
114	58	24	6
202	**88**	**30**	**6**

4.4 Successive differences are not useful for everything.

What if we apply this to the geometric sequence above?

4			*Completing the*
12	8		*table is*
36	24	16	*not necessary.*
108	72	48	*Look at the*
324	216	144	*growth*
972	648	432	

- Note that each successive column grows just as quickly as the first.
- Divide the first column by 4, second by 8, *etc.*, and what happens? The columns are the same.
- Successive differences of a geometric sequence still are geometric sequences.

4.5 An application where successive differences work, amazingly.

- Will return to the "number patterns" examples in the future.
- Skipping to the "figurate numbers" as another example of successive differences.
- Also to define common terminology.

For the terminology, consider the following table header from the context of sequences:

n	T_n	S_n

- In general, n in mathematics is an integer that counts something.
- Here, the term (individual number) within a sequence (list of numbers).

- $n = 1$ is the first term, $n = 2$ the second, *etc.*
- A sequence often is named with a letter. Here T and S for triangular and square. Will explain the names in a moment.
- A particular term n in sequence T is T_n.

To explain the names, start with two points. Draw triangles off of one, squares off the other. Fill in the following table:

n	T_n	S_n
1	1	1
2	3	4
3	6	9
4	10	16
5	15	25

The text provides formula. Plug in n, get a number. Or apply successive differences:

n	S_n	$\Delta_n^{(1)} = S_n - S_{n-1}$	$\Delta^{(2)} = \Delta_n^{(1)} - \Delta_{n-1}^{(1)}$
1	1		
2	4	3	
3	9	5	2
4	16	7	2
5	25	9	2

Terminology notes: A superscript with parenthesis often indicates a step in a process. Here it's the depth of the difference. And Δ (Greek D, "delta") is a traditional letter for differences.

4.6 Next time: Problem solving techniques.

4.7 Homework

Practice is absolutely critical in this class.

Groups are fine, turn in your own work. Homework is due in or before class on Mondays.

- Exercises for Section 1.1:
  - Even problems 2-12. One short sentence of your own declaring *why* you decide the reasoning is inductive or deductive. Feel free to scoff where appropriate.
- Explain why the Section 1.1's example of "2, 9, 16, 23, 30" is a trick question.

- Exercises for Section 1.2:
  - Problems 2, 9, and 10.
  - Problems 14 and 16.
  - Problems 29 (appropriate formula is above problem 21), and 30.
  - Problems 32, 39, and 51.

Note that you *may* email homework. However, I don't use Microsoft™ products (*e.g.* Word), and software packages are notoriously finicky about translating mathematics.

If you're typing it (which I advise just for practice in whatever tools you use), you likely want to turn in a printout. If you do want to email your submission, please produce a PDF or PostScript document.

Chapter 5

Notes for 22 August

Notes also available as PDF.

5.1 The problem solving section is important enough for a full class

- Will dive in on Monday; today is partially review and constructing a bridge to formulas.
- Please read it and look back at the problems we have been solving.
- Think about how the principles apply.
- They may help with the homework.

5.2 Review successive differences: a tool for inductive reasoning on sequences

- Problems to find often use inductive reasoning.
- **Always take care with your premises**. Be sure you understand the framework before exploring with guesses.

Accumulated terminology:

Inductive making an “educated” guess from prior observations.

Sequence list of numbers

Term one of the numbers in a list (in the context of sequences, often used for similar concepts in other contexts)

Arithmetic Sequence defined by an initial number and a constant increment.

Geometric Sequence defined by an initial number and a constant multiple.

- Successive differences to reduce a polynomial sequence to an arithmetic one.
- The last column provides the increment.
- To obtain the next term, fill in the table from the right.
- Not useful for geometric sequences; they remain geometric.

Example, text's problem 4 (not assigned):

n	A_n	$\Delta_n^{(1)} = A_n - A_{n-1}$	$\Delta_n^{(2)} = \Delta_n^{(1)} - \Delta_{n-1}^{(1)}$	$\Delta_n^{(3)}$
1	1			
2	11	10		
3	35	24	14	
4	79	44	20	6
5	149	70	26	6
6	251	102	32	6
7	**391**	**140**	**38**	**6**

5.3 Moving from a table to a formula

Some people work better with formulas. The following is **not** in the text that I can see; this material is extra and meant to be helpful.

Also, this relates inductive and deductive reasoning. The derivation is deductive in breaking down problems and applying rules. But it also is inductive in *how* we chose to break the problem apart.

5.4 Starting point

The goal is

- a formula for $\Delta_n^{(1)}$.

What do we have?

- The relationships in the successive differences table.
- More specifically, that $\Delta^{(2)}$ is an arithmetic sequence starting at 14 with an increment of 6.

Extra mathematics and terminology we need include

- symbols for relations ($<$ is less than, $\geq$ is greater than or equal to) and summations ($\Sigma_{k=i}^{j} = i + (i+1) + \ldots + j$);
- properties of arithmetic: *commutative*, *associative*, *distributive*; and
- that a *series* is the sum of a sequence.

We will prove that

- the formula for $\Delta_n^{(1)}$ is $3n^2 - n = n(3n-1)$.

Continuing the same technique would show that

- the formula for A_n is $n^3 + n^2 - 1$.

Note that each formula involves n^k where k is the number of columns to the right.

5.5 The plan for deriving a formula

1. Rephrase the problem to include what we know.
2. Express the base sequence $\Delta^{(2)}$ as a simple formula.
3. Substitute $\Delta^{(2)}$ into an expression for $\Delta^{(1)}$.
4. Break the resulting complicated expression into simpler components.
5. Pull the pieces back together.
6. Check the result.

5.6 The derivation

5.6.1 Rephrasing the problem

From the definitions used to form the table we know the following *recurrence relationships* hold:

$$\Delta_n^{(1)} \& = \Delta_{n-1}^{(1)} + \Delta_n^{(2)}, \text{ and}$$
$$\Delta_n^{(2)} \& = \Delta_{n-1}^{(2)} + \Delta_n^{(3)}.$$

5.6.2 Expressing the base sequence

$\Delta^{(3)}$ is a constant sequence for $n \geq 4$, so we know that $\Delta^{(2)}$ is an arithmetic sequence starting with 14 and using 6 as its increment. We can express the nth term as

$$\Delta^{(2)}_n = 14 + (n-3) \cdot 6 \text{ for } n \geq 3$$

and extend it to the previous entries by defining

$$\Delta^{(2)}_n = 0 \text{ for } n < 3.$$

(Note: Multiplication is written many ways. Each of $a * b = a \times b = a \cdot b = ab$ are different common forms. The $\times$ symbol can be confused with the letter x and is not used often.)

(The term $n - 3$ shifts n down so the sequence fits our tables. We could have built the table directly across with $\Delta^{(1)}_n = A_{n+1} - A_n$. Either choice is fine, and this alternative likely work more clearly here.)

5.6.3 Substituting into $\Delta^{(2)}$ into the expression for $\Delta^{(1)}$

Expanding the recurrence for $\Delta^{(1)}_n$ provides

$$\Delta^{(1)}_n = 10 + \Sigma^n_{k=2}\Delta^{(2)}_k \text{ for } n \geq 2,$$

and again we define $\Delta^{(1)}_n = 0$ for $n < 2$.

(The expression $\Sigma^j_{k=i} x_n$ denotes the sum of all terms starting at i and ending after j. Σ is a capital Greek S. So $\Sigma^5_{k=1} i = 1 + 2 + 3 + 4 + 5 = 15$.)

Consider only the term $\Sigma^n_{k=2}\Delta^{(2)}_k$. First note that $\Delta^{(2)}_n$ is non-zero only when $n \geq 3$, so we can pull out the $k = 2$ term,

$$\Sigma^n_{k=2}\Delta^{(2)}_k = 0 + \Sigma^n_{k=3}\Delta^{(2)}_k.$$

Then simplify using $0 + x = x$ and substitute the expression for $\Delta^{(2)}_n$,

$$\Sigma^n_{k=2}\Delta^{(2)}_k = \Sigma^n_{k=3}(14 + 6(k-3)).$$

5.6.4 Breaking down the complicated expression

Now we use a few properties of addition and multiplication. We will return to these definitions in later chapters.

- Addition is *commutative*, $a+b = b+a$, and *associative*, $(a+b)+c = a+(b+c)$.
- Multiplication also is commutative, $ab = ba$, and associative, $(ab)c = a(bc)$.

- Multiplication is the same as repeated adding, so $3x = x + x + x$.
- Multiplication is *distributive* over addition, so $ab + ac = a(b + c)$.

With the associative and commutative properties of addition, we rewrite

$$\Sigma_{k=3}^{n}(14 + 6(k-3)) = (\Sigma_{k=3}^{n}14) + (\Sigma_{k=3}^{n}6(k-3)).$$

Again, we break the sum apart and work on the pieces. Because multiplication and repeated addition are the same,

$$\Sigma_{k=3}^{n}14 = 14 \cdot ((n-3)+1) = 14 \cdot (n-2).$$

There are $j - i + 1$ terms in the *series* $\Sigma_{k=i}^{j}14$. A *series* is the sum of a *sequence*.

Applying the distributive property,

$$\Sigma_{k=3}^{n}(k-3) \cdot 6 = 6 \cdot \Sigma_{k=3}^{n}(k-3),$$

where we pull out the 6 because it does not depend on the summation variable k. Applying associativity and commutativity again to the $\Sigma_{k=3}^{n}(k-3)$ term,

$$\Sigma_{k=3}^{n}(k-3) \cdot 6 = 6 \cdot (-3(n-2) + \Sigma_{k=3}^{n}k).$$

Consider the term $\Sigma_{k=3}^{n}k$. We know the sum from 1 to n is $n(n+1)/2$. We present two routes for reducing $\Sigma_{k=3}^{n}k$ to what we already know. The first is to extend the series and subtract the added terms, so

$$\Sigma_{k=3}^{n}k = (\Sigma_{k=1}^{n}k) - \Sigma_{k=1}^{2}k = n(n+1)/2 - 3.$$

The second shifts the summands to the summation starts at 1. It's far more complicated but also more general. **I won't cover this during class, but it's in the notes.**

The other route, shifting the summation

Our reason for exploring this route is to demonstrate shifting the indices over the summation. To do this, we need to substitute $k = i + 2$ to reach

$$\Sigma_{k=3}^{n}k = \Sigma_{i+2=3}^{n}(i+2).$$

Now remember that the Σ notation implies that we use $i + 2 = 3, 4, \ldots, n$. To more the 2 across the equality, we must subtract it from all of the indices, and

$$\Sigma_{k=3}^{n}k = \Sigma_{i+2=3}^{n}(i+2) = \Sigma_{i=1}^{n-2}(i+2).$$

Now we can separate the terms again and apply $\Sigma_{i=1}^{n-2} i = (n-2)(n-1)/2$ as well as $\Sigma_{i=1}^{n-2} 2 = 2(n-2)$ to see that

$$\begin{aligned}\Sigma_{k=3}^{n} k &= (n-2)(n-1)/2 + 2(n-2)\\ &= (n-2)(n-1+4)/2\\ &= (n-2)(n+3)/2.\end{aligned}$$

This appears to be a different result, but subtracting one expression from the other and expanding results in zero and proves that they are equal.

5.6.5 Pulling the pieces together

To recap, we began with the relationships

$$\Delta_n^{(2)}\& = 14 + 6(n-3) \text{ for } n \geq 3, \text{ and}$$
$$\Delta_n^{(1)}\& = 10 + \Sigma_{k=2}^{n} \Delta_k^{(2)} \text{ for } n \geq 2.$$

Substituting $\Delta_n^{(2)}$ into $\Delta_n^{(1)}$, regrouping the result and expanding produced many non-trivial subexpressions. Gathering them into one shows

$$\Delta_n^{(1)} = 10 + (14(n-2) + 6(-3(n-2) + n(n+1)/2 - 3)) \text{ for } n \geq 2.$$

Simplifying reveals

$$\Delta_n^{(1)} = 10 + (3n^2 - n - 10) = 3n^2 - n \text{ for } n \geq 2.$$

5.6.6 Checking the result

n	$\Delta_n^{(1)}$	$3n^2 - n$
1		
2	10	12-2 = 10
3	24	27-3 = 24
4	44	48-4 = 44
...	...	...
10		300-10 = 290

5.7 Homework

Practice is absolutely critical in this class.

Groups are fine, turn in your own work. Homework is due in or before class on Mondays.

- Exercises for Section 1.2:
  - Problem 49. Working inductively here is far simpler than deriving the formula.
  - Problems 51, 54. Try both inductively **and** by playing with the formula.

Note that you *may* email homework. However, I don't use Microsoft™ products (*e.g.* Word), and software packages are notoriously finicky about translating mathematics.

If you're typing it (which I advise just for practice in whatever tools you use), you likely want to turn in a printout. If you do want to email your submission, please produce a PDF or PostScript document.

Chapter 6

Solutions for first week's assignments

Also available as PDF.

6.1 Notes on received homeworks

- The goal of homeworks is practice on the topics covered in the text and in class. If you're unsure how to tackle one problem, look at the problems nearby or at examples. One may be more clear to you and help you with the assigned problem.
- I have office hours now. Monday and Wednesday 1.30pm to 2.30pm (or possibly later) in the Math Lab down the hall.
- Be sure to read the **entire** problem. Many submissions contained only partial answers even when it was clear you understood the mechanism.
- With problems involving large numbers, expect most calculators and computer software to break. Try to *check* results using properties of the input numbers. For example the product of two numbers with units digit 1 also has units digit 1. Or that the product of two d digit numbers has either $2d$ or $2d-1$ digits. (Think about long-hand multiplication to find these and other properties.)
- If there are questions about which problems were assigned or what the problem is asking, contact me even if it's the night before the homework is due! I may not respond instantly, but it's worth a shot.
- Because there was apparent confusion over which problems were assigned, I will start providing the homework on a separate page as well as directly

in the notes.

- In general, writing out steps cushions the blow if the result is incorrect. And writing out *reasons* helps even more. If your homework must be late, reasoning in your own style and words shows you did not just copy solutions. *This class is as much about the method of thinking and communicating as it is about the final results!*
- Remember that homework is one 20% chunk. But there will be 14 or 15 assignments. Each is at *most* ... And if there are 10-20 problems per assignment, then each assignment is at most... This is another reason why homeworks are frequent. The impact of each assignment is a little less when there are many.

6.2 Exercises for Section 1.1

6.2.1 Even problems, 2-12

2 Deductive. The "if-then" rule about medicine is a premise that is immediately applied.

4 Inductive. The three children are examples, but there is no rule dictating birth gender.

6 Deductive. An "if-then" rule is given and applied.

8 Deductive. The rule is implicit, but the conclusion is derived from data and rules rather than repeated examples.

10 Inductive. Only repeated observations are used to justify the conclusion.

12 Inductive. Again, only observations enter into the reasoning.

6.3 Explain the "trick" of Section 1.1's example

A list of numbers as in Section 1.1 (2, 9, 16, 23, 30) does not mean anything on its own. The context before this example implies that one should look for an arithmetic relationship.

The "trick" is that a premise is withheld. As in poorly written mystery novels, crucial information is not available.

All reasoning is based on premises (hypotheses, suppositions, *etc.*) wether implicit or explicit. "Trick" questions like Section 1.1's example rely on misleading you into using an incorrect implicit premise.

6.4 Exercises for Section 1.2

6.4.1 Problems 2, 9, and 10

Problem 2

3		
14	11	
31	17	6
54	23	6
83	29	6
118	35	6
159	**41**	**6**

Problem 9

The formula provided in the text is of order 4, or in other words the highest power of the argument n is n^4. (Another phrase for this is that the formula is *quartic.*) We expect to need 4 columns to the right of the original sequence (1, 2, 4, 8, 16, 31) to reach an arithmetic sequence.

points	regions	$\Delta^{(1)}$	$\Delta^{(2)}$	$\Delta^{(3)}$	$\Delta^{(4)}$
1	1				
2	2	1			
3	4	2	1		
4	8	4	2	1	
5	16	8	4	2	1
6	31	15	7	3	1
7	$\boxed{\mathbf{57}}$	**26**	**11**	**4**	**1**
8	$\boxed{\mathbf{99}}$	**42**	**16**	**5**	**1**

The formula provided is

$$R(n) = \frac{1}{24}\left(n^4 - 6n^3 + 23n^2 - 18n + 24\right).$$

One can compute this directly with any method to verify that the answer is **99**.

One convenient way to rewrite a polynomial for evaluation is *Horner's rule.* Horner's rule applies the distributive property of multiplication over addition to pull factors of n out of subexpressions. This rule not only is faster when using a calculator, it also incurs fewer rounding errors when n is not an integer.

Applying Horner's rule,

$$R(n) = \frac{1}{24}((n^3 - 6n^2 + 23n - 18)n + 24)$$
$$R(n) = \frac{1}{24}(((n^2 - 6n + 23)n - 18)n + 24)$$
$$R(n) = \frac{1}{24}((((n - 6)n + 23)n - 18)n + 24)$$

Subsituting 8 we find that

$$\begin{aligned} R(8) &= \frac{1}{24}((((8-6)\cdot 8 + 23)\cdot 8 - 18)\cdot 8 + 24) \\ &= \frac{1}{24}(((2\cdot 8 + 23)\cdot 8 - 18)\cdot 8 + 24) \\ &= \frac{1}{24}((39\cdot 8 - 18)\cdot 8 + 24) \\ &= \frac{1}{24}(294\cdot 8 + 24) \\ &= \frac{1}{24}(2376). \end{aligned}$$

Dividing directly again verifies the result is 99, but a technique to avoid the division is recognizing that $2376 = 2400 - 24$. Then

$$R(8) = \frac{1}{24}(2376) = \frac{1}{24}(2400 - 24) = 100 - 1 = 99.$$

Problem 10

The problem is of order 2 (or is *quadratic*), so we expect two columns beyond the initial sequence.

n	$n^2 + 3n + 1 = (n+3)n + 1$	$\Delta^{(1)}$	$\Delta^{(2)}$
1	5		
2	11	6	
3	19	8	2
4	29	10	2
5	**41**	**12**	**2**

Substituting 5 into $(n+3)n+1$ produces $(5+3)\cdot 5+1 = 8\cdot 5+1 = 41$, verifying the result.

6.4.2 Problems 14 and 16

Problem 14

There are two reasonable ways to extend the left pattern. Either is reasonable, and both demonstrate the same property.

The first prepends 10 to each number on the left. The resulting pattern is

$$\begin{aligned} 101 \times 101 &= 10\,201, \\ 10\,101 \times 10\,101 &= 102\,030\,201, \text{ and} \\ 1\,010\,101 \times 1\,010\,101 &= 1\,020\,304\,030\,201. \end{aligned}$$

The second possiblity "reflects" the number across the leading or trailing 1. The resulting pattern is

$$\begin{aligned} 101 \times 101 &= 10\,201, \\ 10\,101 \times 10\,101 &= 102\,030\,201, \text{ and} \\ 101\,010\,101 \times 101\,010\,101 &= 10\,203\,040\,504\,030\,201. \end{aligned}$$

The common property is that squaring a number with alternating 1 and 0 digits

With these short sequences, the zeros only serve to make the pattern more obvious. Note that $11^2 = 121$ and $111^2 = 12\,321$. This pattern will break after the central digit is 9. Why?

*Note that computing $101\,010\,101^2$ with common desktop computers may produce 10 203 040 504 030 200. The last digit falls off the end of how computers represent floating-point numbers. Computing in integers on "32-bit" computers may produce **28** if the calculation wraps around the 32-bit boundary.*

This is one reason why looking for patterns and developing a number sense is important. Errors in calculated results depend on the method used for calculation. Most programs or devices do not explain their methods, so recognizing patterns and other properties are important to prevent being misled.

Problem 16

The next line could well be

$$1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1 = 5^2.$$

One method for verifying the result is simple calculation.

Another is to rearrange the problem slightly to show the pattern

$$\begin{aligned}
1 &= (\sum_{i=1}^{1} i) + (\sum_{i=1}^{1-1} i) = 1^2, \\
1+2+1 &= (\sum_{i=1}^{2} i) + (\sum_{i=1}^{2-1} i) = 2^2, \\
1+2+3+2+1 &= (\sum_{i=1}^{3} i) + (\sum_{i=1}^{3-1} i) = 3^2, \text{ and} \\
1+2+3+4+3+2+1 &= (\sum_{i=1}^{4} i) + (\sum_{i=1}^{4-1} i) = 4^2.
\end{aligned}$$

Using the formula $\sum_{i=1}^{n} i = n(n+1)/2$, the n^{th} middle form is

$$\frac{n(n+1)}{2} + \frac{(n-1)n}{2} = \frac{n^2+n+n^2-n}{2} = n^2.$$

So the fifth term is indeed 5^2.

6.4.3 Problems 29 and 30

Problem 29

There are two clear ways to extend the formula $S(n) = n(n+1)/2$ into a formula for the sum $2+4+6+\cdots+2n$.

One is to recognize that $2+4+6+\cdots+2n = 2(1+2+3+\cdots+n) = 2S(n) = n(n+1)$. Rephrasing the original problem using summation notation, we have used $\sum_{i=1}^{n} 2i = 2\sum_{i=1}^{n} i = n(n+1)$.

Another is to consider the sum $2+4+6+\cdots+2n = (1+1)+(2+2)+(3+3)+\cdots+(n+n) = (1+2+3+\cdots+n)+(1+2+3+\cdots+n) = S(n)+S(n) = 2S(n) = n(n+1)$. In summation notation, $\sum_{i=1}^{n} 2i = \sum_{i=1}^{n}(i+i) = (\sum_{i=1}^{n} i)+(\sum_{i=1}^{n} i) = n(n+1)$.

Problem 30

This problem is about phrasing mathematical problems in a way that respects the order of operations.

Some possibilities are the following:

- The square of the sum of the integers from 1 to n equals the sum of the cubes of those same numbers.

- The square of the sum of the first n whole numbers is the sum of their cubes.
- Squaring the sum of the integers from 1 to n is the same as summing their cubes.

The key is that on the left a sum is squared, while on the right cubes are added.

6.4.4 Problems 32, 39, and 51

This sequence of problems demonstrates similar points in different ways. They are all related to each other and to Problem 16.

Problem 32

Each column (or row) of the blocked triangles represents the integers 1, 2, 3, and 4 by the number of blocks in the column (or row). The total number of blocks in each triangle is 1+2+3+4.

When flipped and combined, the total number blocks is the sum of the two triangles, or $2 \cdot (1 + 2 + 3 + 4)$. The combined figure is a rectangle consisting of $4 \cdot 5$ blocks. So $2(1 + 2 + 3 + 4) = 4 \cdot 5$, or $1 + 2 + 3 + 4 = (4 \cdot 5)/2$.

A better diagram would replace the left-most arrow with an addition operator (+).

Problem 39

Drawing out the dots demonstrates the solution directly.

Or use the result of Problem 16. Note that the first sum, $\sum_{i=1}^{n} i$, is the n^{th} triangular number and that the second sum, $\sum_{i=1}^{n-1} i$, is the $(n-1)^{\text{th}}$ triangular number. Thus Problem 16 demonstrated that the sum of two consecutive triangular numbers is a square.

Problem 51

Either draw a few consecutive figures from Problem 39 or use Problem 16.

6.4.5 Problem 49

The first pattern to observe is they are all of the form $p(n)/2$, where $p(n)$ is some polynomial of n. The next pattern is that the numerator $p(n) = n(a \cdot n - b)$ for integers a and b. Then both a and b increase by one when adding a side.

Using these patterns, the n^{th} nonagonal number is

$$N(n) = \frac{n(7n-5)}{2}.$$

Substituting 6, $N(6) = 6(7 \cdot 6 - 5)/2 = 3(42 - 5) = 111$, adding more evidence to the conjecture.

6.4.6 Problems 51 and 54

Problem 51

Oops. I think meant to give Problem 52 rather than repeat Problem 51, but that's my fault.

Problem 54

Filling in a few values,

n	$T(n-1)$	$3T(n-1)+n$
2	1	5
3	3	12
4	6	22
5	10	35

The first, 5, suggests the five-sided pentagon that produces the second pentagonal number. Later numbers add additional evidence.

One could prove the relationship by expanding $3T(n-1)+n$ and simplifying the expression, or

$$\begin{aligned}
3T(n-1)+n &= 3\left(\frac{(n-1)(n-1+1)}{2}\right)+n \\
&= \frac{3n^2-3n}{2}+n \\
&= \frac{3n^2-3n+2n}{2} \\
&= \frac{3n^2-n}{2} \\
&= \frac{n(3n-1)}{2} \\
&= P(n).
\end{aligned}$$

Chapter 7

Notes for 25 August

Notes also available as PDF.

7.1 Problem solving principles

- So far we've covered problem solving by recognizing and playing with patterns.
- Pattern matching is one part of mathematical "common sense" and a valuable way to start on a problem.
- Recognizing patterns is only one approach to problem solving.
- Mathematicial George Pólya spent much of his life considering how mathematicians and others approach problems.
- His problem solving principles are a good general description.

7.1.1 Pólya's principles

These are **principles** and not a recipe or a plan. Use these to *form* a problem-solving plan. (Problem solving itself is a problem. . .).

The principles are very well explained in Pólya's light book, How to Solve It. He later goes into much depth in his two part series on Mathematics and Plausable Reasoning (volume 1, volume 2).

- Understand the problem
- Divise a plan
  - Text provides ideas, and we will cover a few.

- Carry out the plan
- Examine the solution

7.1.2 Understand the problem

- Goals in understanding: Find the data you have and the solution you need.
- Often helps to rephrase a few ways.
  - In English (or whatever is appropriate)
  - With mathematical notation
- Determine what may be relevant.
- Sketch the problem graphically, with numbers, with physical items... Whatever works for **you** on this problem.
- Decide if the problem may **have** a reasonable solution.
- Example: Problem of the seven coins.

7.1.3 Divise a plan

- Goal: Decide on an approach to the problem.
  - The first plan is your initial plan.
  - You may need to toss it aside and form a new one...
  - Often useful to approach a problem with a few plans at once.
- Taxonomy of plans in the text.
- Plans apply inside and in conjunction with other plans.
- Few problems fall to a single technique.
- Being excessively clever is not a plan.
- Most "cleverness" comes from experience and practice.

  Sure I'm lucky. And the more I practice, the luckier I get. – Gary Player, golfer
- When I think about the strategy, I consider what I want to write about it. A major goal of mathematics is communicating ideas well.
- To communicate a plan to others, you need all the understanding and precision necessary to carry out the plan.

7.1.4 Carry out the plan

- Details depend on the plan…
- May work, may not work.
- Dead-ends are common when approaching new styles of problems.
- Moving through the plan requires attention to detail.
  - Mathematicians and scientists grow to **loath** +/- signs.
  - Eventually, you learn which details can be “fixed” later.

7.1.5 Examine your solution

- Very, very important.
- Check your results somehow, possibly by varying the problem a little.
- Trying a different solution technique also can check your problem.
  - Re-trying the same technique often does *not* help. People often make the same mistakes.
- Try to generalize a little.
- **Interpret** your results by writing sentences. Often provides a check in itself, or leads to an alternate route.
  - Common in mathematics: First publication of a result is long and hairy. Interested people being interpreting it, and a short or more direct proof is found.
  - Erdős and the “book proof”.

7.2 Making a lists and tables

- Lists are useful for
  - counting items, and
  - systematically searching possibilities.
- Planning includes deciding how to construct the list.
  - The method must be systematic and easy.
  - Will get plenty of practice when building logic tables.
- *Careful* planning helps to construct a *smaller* list.
  - Relationships found from understanding and guessing can help.

- We will talk about bisection, working with a list but not forming all of it, next time ("trial and error").

7.2.1 Example of a table

How many ways can you form 21 cents from dimes, nickles, and pennies?

While thinking of the problem, note that the last 1 cent does not change the number of ways to form the total. There always will be one penny involved. We should just drop that one penny.

Plan: Form a table. Then the plan becomes *how* to form the table.

We can start with an extreme solution and modifiy it one row at a time. In the table, we push change from left (higher) to right, while checking the total.

# dimes	# nickles	# pennies	total cents = 20
2			2*10 = 20
1	2		10+2*5 = 20
1	1	5	10+5+5*1 = 20
1		10	10+10*1 = 20
	4		4*5 = 20
	3	5	3*5+5*1 = 20
	2	10	2*5+10*1 = 20
	1	15	1*5+15*1 = 20
		20	20*1 = 1

So there are nine ways of forming 21 cents from dimes, nickles, and quarters.

7.3 Searching by guessing

- Guessing, a good start to just about every problem. Helps to:
  - find examples,
  - discover relationships,
  - gain a feel for the problem,
  - or just find the answer.
- Guessing randomly is of little use.
- Use relationships gleaned from understanding the problem to prune your guesses.

- Sometimes the relationships are as easy as “smaller inputs yield smaller results.”

7.3.1 Example for guessing and checking

Complete the following triangle such that the numbers in the vertices are equal to the sum of the variables adjacent to them. Assume all the variables are positive integers.

$$\begin{array}{ccccc} & & 16 & & \\ & a & & b & \\ 11 & & c & & 15 \end{array}$$

- When considering the problem, look for relationships that can guide your guesses.
- Because a, b, and c are positive, we know the sum of any two is greater than either. That is, $16 = a + b > a$, and $16 = a + b > b$.
- The initial plan becomes to pick numbers less than the ones shown.
- Try a guess, and notice that you only need to pick **one** number. The rest are completely determined.
- So you can start with a and guess from numbers less than 11.

$$\begin{array}{ccccc} & & 16 & & \\ & \mathbf{6} & & \mathbf{10} & \\ 11 & & \mathbf{5} & & 15 \end{array}$$

Now look back and consider some new relationships:

- $16 + 11 + 15 = 2(6 + 5 + 10)$.
- Try other numbers in the vertices, see if this relationship holds.
- For which numbers does this problem have a solution when a, b, and c all are positive integers?
  - What are the smallest numbers?
  - What property must the sum of the numbers have?

7.4 Understanding dependencies, or ”working backward”

- “Working backward” is short-hand for following the chain of dependencies in a problem.

- Useful when it looks like there's no where to start, but there are definite known points along the way.
- The plan:
  - Start with what you know.
  - Derive every quantity you can from that data.
  - Repeat.

7.4.1 Example for following dependencies

Example 2 from the text:

> Rob goes to the racetrack on a weekly basis. One week he tripled his money but then lost \$12. Returning to the track the next week with all his money, he doubled his money but then lost \$40. Again he returned to the track with his money. He quadrupled his money and lost nothing, taking home \$224.
>
> How much money did take on his first week above?

First, rephrase the problem mathematically. Let M_n be his total starting in week n. We want M_1. From the problem,

$$\begin{aligned} M_2 &= 3M_1 - 12, \\ M_3 &= 2M_2 - 40, \text{ and} \\ M_4 &= 4M_3 = 224. \end{aligned}$$

As written, M_2 depends on M_1 and so on. But we only have the *last* total, M_4.

So our plan:

- Rearrange dependencies to start from what we have and lead to what we want.

Thus,

$$\begin{aligned} M_1 &= (M_2 + 12)/3, \\ M_2 &= (M_3 + 40)/2, \text{ and} \\ M_3 &= M_4/4. \end{aligned}$$

Substituting $M_4 = 224$,

$$\begin{aligned} M_3 &= 224/4 = 56, \\ M_2 &= (56 + 40)/2 = 48, \text{ and} \\ M_1 &= (48 + 12)/3 = 20. \end{aligned}$$

Looking back:

- Alternate approach: We could have algebraically substituted M_2 and M_3 into the expression for M_4 and solved.
- End result is the same, but with less algebra.

7.5 Next time: more techniques

7.6 Homework

Practice is absolutely critical in this class.

Groups are fine, turn in your own work. Homework is due in or before class on Mondays.

- Exercises for 1.3
  - Understanding the problem: Problem 6
  - Guessing and checking: problem 12
  - Listing: problems 31, 35
  - Dependencies and diagramming: 28, 57

Note that you *may* email homework. However, I don't use Microsoft^{TM} products (*e.g.* Word), and software packages are notoriously finicky about translating mathematics.

If you're typing it (which I advise just for practice in whatever tools you use), you likely want to turn in a printout. If you do want to email your submission, please produce a PDF or PostScript document.

Chapter 8

Notes for 27 August

Notes also available as PDF.

8.1 Review: Pólya's problem-solving principles

These are **principles** and not a recipe or a plan. Use these to *form* a problem-solving plan. (Problem solving itself is a problem. . .).

- Understand the problem
  - Determine what data you have and what quantities you need.
  - Try rephrasing the problem in words and in symbols.
  - Look for relationships: symmetries, which items are fully determined by others, *etc.*
- Divise a plan
  - Remember previous similar problems and extend the pattern.
  - Use relationships to help select possible plans.
  - Make plans as specific as possible.
- Carry out the plan
  - Pay attention to details.
- Examine the solution
  - Check your solution, either directly or against relationships.
  - Consider related (and future) problems.

Previous tactics:

- Making lists or tables:
  - Good for systematic searching or counting.
  - Requires care in forming the tables.
- Guessing:
  - Good for discovering or extending relationships.
  - Rarely a complete plan on its own, but great for starting.
- Dependencies, or working backwards:
  - Essentially, find the starting point and then exhaust all the rules or formulas that apply. (Breadth-first)
  - Or try following one piece of data through all possible rules, then backtrack. (Depth-first)

Today, a few more tactics:

- "Trial and error", but a bit more systematically and quickly through bisection.
- Using simpler sub-problems to find patterns.

8.2 Effective trial and error by bisection

Remember:

Geometric sequence Sequence of numbers defined by a starting number and a constant multiplier. The second number is generated by multiplying by the constant, the third by multiplying again, and so on.

Consider the sequence where 3 is the starting number and two is the constant.

n	Term	
1	3	$= 3 \cdot 2^0 = 3 \cdot 2^{n-1}$
2	6	$= 3 \cdot 2^1$
3	12	$= 3 \cdot 2^2$
$\vdots$	$\vdots$	$\vdots$

Which term in the sequence is 768?

8.2.1 Understanding the problem

- What do we have?
  - Definition of a specific geometric sequence, $3 \cdot 2^{n-1}$.

- What do we need?
  - A specific term equal to 768.

8.2.2 Forming plans

- Baring use of logarithms, what is one possible plan?
  - Extend the table until it reaches 768.
- A list would work, but could be long.
  - Guess how long.
    - What is 2^{10}? *1024*. What is $3 \cdot 2^{10}$? *3072*.
- Now we have *more information*.
  - So we know the third term is smaller, $12 < 768$, and the eleventh term is larger, $3072 > 768$.
- Consider filling in only *some* entries of the list. Which to try next?
- Try going half-way, a new plan.

8.2.3 Carrying out the new plan

- Half-way is the seventh term, $3 \cdot 2^6 = 192$.
- Now what do we know?
  - That 768 must be after the seventh term and before the eleventh.
- Half-way again is the ninth term, $\mathbf{3 \cdot 2^8 = 768}$.

8.2.4 Looking back

- Calculated *three* additional terms (12, 7, 9) rather than six (4, 5, 6, 7, 8, 9).
- When considering finding an entry by a list, look for an ordering relationship.
  - Each entry no smaller or larger than the previous. (*How does that differ from always being larger/smaller?*)
  - Or if you're looking for a property, try to order the list so that all those after a point have (or don't have) that property.
- Use the ordering to reduce work (and errors).

- Calculate a few entries spaced far apart.
- Find entries that *bracket* your search target. So you know your target is between a and b, or in (a, b).
- Then look half-way to form a new bracket. The new bracket will be one of $(a, (a + b)/2)$ or $((a + b)/2, b)$. Remember to round consistently.

8.3 Simpler sub-problems for finding patterns

What is the units digit of 7^{100}?

- Relatively straight forward, so try examples to gain a feel for the problem.

Number	Expanded	Last digit
7^0	1	1
7^1	7	7
7^2	49	9
7^3	343	3
7^4	2401	1
7^5		7
7^6		9
$\vdots$	$\vdots$	$\vdots$

- Want a quick plan for generating examples.
  - Bisection doesn't apply. We know which entry we want, and the digits go up and down.
  - What relationship can we use?
  - *While generating* 2401*, only needed the product of 7 with the prior last digit.*
- But now we have 1 as the last digit. A few more lines, and we see the pattern. The 1 begins a pattern.
- What is the pattern?
  - The table breaks into groups of four lines.
  - The all repeat 7, 9, 3, 1.
- Restate what we have more abstractly. For each i, we know that
  - the last digit of $7^{4i} = 1$,
  - the last digit of $7^{4i+1} = 7$,
  - the last digit of $7^{4i+2} = 9$, and
  - the last digit of $7^{4i+3} = 3$.

- Where does 7^{100} fall?
  - $7^{100} = 7^{4 \cdot 25}$, so its last digit is 1.

Looking back:

- On the technical side, we only needed to track the last digit.
  - This will return when we break apart numbers in Chapters 4-6.
- There were no simple phases corresponding to the principles, but all applied.
- New facts and new patterns let us understand more of the problem.
- Each led to new plans and new phrasings.

8.4 Other sources for tactics and examples

- The Math 202 notes. The notes currently are at

  http://jriedy.users.sonic.net/math202-f08/ .
- Pólya's books. In particular, the majority of *How to Solve It* consists of a compendium of ideas and techniques.

8.5 Next time: Reading graphs and charts

Recommended reading: Anything by Edward Tufte. The "Ask E.T." section of his personal site (http://www.edwardtufte.com) has examples of excellent and poor graphics.

8.6 Homework

Practice is absolutely critical in this class.

Groups are fine, turn in your own work. Homework is due in or before class on Mondays.

- Describe not only the result but also your approach in the following.
- From problem set 1.3:
  - Problem 28 for following dependencies
  - Problem 40 both for bisection and guessing a reasonable range
  - Problem 52, think of bisection

  - Problem 56, think about lines and the shapes you can form
  - Problem 61, look for a pattern

- How many ways can you make change for 60 cents using pennies, nickles, dimes, and quarters. Either take great care in forming a long list, or look for a relationship using smaller problems.

  A hint for a long list: Do you need to move pennies one at a time?

  A hint for a relationship: Consider the old example of 20 cents using pennies, nickles and dimes. How many ways are there to change 20 cents using only pennies and nickles? How many ways to change 20 cents minus one dime using all the coins? The relationship makes constructing a table much easier.

Note that you *may* email homework. However, I don't use Microsoft™ products (*e.g.* Word), and software packages are notoriously finicky about translating mathematics.

If you're typing it (which I advise just for practice in whatever tools you use), you likely want to turn in a printout. If you do want to email your submission, please produce a PDF or PostScript document.

Chapter 9

Notes for 29 August

Notes also available as PDF.

9.1 Review: Pólya's problem-solving principles

These are **principles** and not a recipe or a plan. Use these to *form* a problem-solving plan. (Problem solving itself is a problem. . .).

- Understand the problem
- Divise a plan
- Carry out the plan
- Examine the solution

Some tactics we've covered:

- Making lists or tables
- Guessing
- Dependencies, or working backwards
- "Trial and error", but a bit more systematically and quickly through bisection
- Using simpler sub-problems to find patterns

9.2 Notes on the homework

- The goal of homeworks is practice on the topics covered in the text and in class. If you're unsure how to tackle one problem, look at the problems nearby or at examples. One may be more clear to you and help you with the assigned problem.
- I have office hours now. Monday and Wednesday 1.30pm to 2.30pm (or possibly later) in the Math Lab down the hall.
- Be sure to read the **entire** problem. Many submissions contained only partial answers even when it was clear you understood the mechanism.
- With problems involving large numbers, expect most calculators and computer software to break. Try to *check* results using properties of the input numbers. For example the product of two numbers with units digit 1 also has units digit 1. Or that the product of two d digit numbers has either $2d$ or $2d - 1$ digits. (Think about long-hand multiplication to find these and other properties.)
- If there are questions about which problems were assigned or what the problem is asking, contact me even if it's the night before the homework is due! I may not respond instantly, but it's worth a shot.
- Because there was apparent confusion over which problems were assigned, I will start providing the homework on a separate page as well as directly in the notes.
- In general, writing out steps cushions the blow if the result is incorrect. And writing out *reasons* helps even more. If your homework must be late, reasoning in your own style and words shows you did not just copy solutions. *This class is as much about the method of thinking and communicating as it is about the final results!*
- Remember that homework is one 20% chunk. But there will be 14 or 15 assignments. Each is at *most* ... And if there are 10-20 problems per assignment, then each assignment is at most... This is another reason why homeworks are frequent. The impact of each assignment is a little less when there are many.

9.3 Reading graphs: delayed until Monday (or later)

9.4 Homework

Practice is absolutely critical in this class.

Groups are fine, turn in your own work. Homework is due in or before class on Mondays.

- Following Pólya's principles, write a careful solution for the following problems:
  - From last week's homework: Section 1.2, problems 9 and 49
  - From this week's homework: Section 1.3, problem 40

  For the write-up, use each of Pólya's principles as a section heading. Begin with a section on **Understanding the Problem** (or an equivalent phrase) detailing what you have, what you want, and what (if any) relationships you see immediately. Then under something like **Devise a Plan**, construct a detailed plan. In **Carry out the Plan**, perform whatever operations are required. Then under **Examine Your Solution** (or Look Back, *etc.*), check your solution and rephrase it in English

- Using whatever calculator or program
  - Compute $1/7$. Write down the number exactly as displayed. Then subtract what you have written from the calculator's or program's result. For a calculator, divide one by seven and then subtract off what you see without storing the result elsewhere. For a spreadsheet or other interface, divide one by seven. Then compute $1/7 - .14\cdots$ for whatever was displayed. **What is the result? What did you expect? What result did others find?**
  - Enter .1 into whatever device you use. Add .1 to it. Repeat eight more times, for a total of $10 \cdot .1$. Subtract 1. **What is the result? What did you expect? What result did others find?**

Note that you *may* email homework. However, I don't use Microsoft$^{\text{TM}}$ products (*e.g.* Word), and software packages are notoriously finicky about translating mathematics.

If you're typing it (which I advise just for practice in whatever tools you use), you likely want to turn in a printout. If you do want to email your submission, please produce a PDF or PostScript document.

Chapter 10

Solutions for second week's assignments

Also available as PDF.

10.1 Exercises for Section 1.3

10.1.1 Problem 6: Understanding

The problem begins by stating that "[t]oday is your first day driving a city bus." At the end, the question posed is "[h]ow old is the bus driver?" The other information is irrelevant, and the result is your age. (You need not *provide* your age. The fact that "you" are the bus driver is all that matters.)

10.1.2 Problem 12: Guessing and checking

When playing around with the problem, you may realize that if each of the three parts has the same sum, then each must equal 1/3 of the total sum. The sum $\sum_{i=1}^{12} i = 12(12+1)/2 = 6 \cdot 13$, so we can guess that three equal parts each will total $6 \cdot 13/3 = 2 \cdot 13 = 26$.

Now the problem becomes positioning lines such that the totals are 26. The smaller numbers (*e.g.* 1, 2) need combined with some of the larger numbers (*e.g.* 11, 12) to total 26. So we can guess that one of the lines will be nearly horizontal across the top. First try a line that separates the top four numbers, so $11 + 12 + 1 + 2 = 26$.

Now the other line cannot intersect the first *inside* the clock or else the clock will be divided into four parts. My first guess was to anchor one end of the new line where the previous line ran under 2. That would cut a pie-like slice and only include numbers on the opposite side. The closest adding the facing side can come to 26 is $10 + 9 + 8 = 27$, so clearly that was not correct. If we chop off the 8, that leaves $10 + 9 = 19$ and needs 7 more to match 26. So run the right-hand side down, no longer cutting out a pie-like slice but including some numbers from both sides. Then we can include $3 + 4 = 7$.

So far, my guesses separated $11 + 12 + 1 + 2 = 26$ and $10 + 9 + 3 + 4 = 26$. With the remaining numbers, $5 + 6 + 7 + 8 = 26$, so the problem is solved.

In hindsight, however, I see my previous guess was being far too fancy. Note that each slightly diagonal slice adds to 13. That is, $12 + 1 = 13$, $11 + 2 = 13$, and so on down to $7 + 6 = 13$. So we need only combine consecutive slices to create the necessary three regions.

10.1.3 Problem 31: Listing

here most of the problem lies in figuring out an appropriate representation. Then starting to make lists show the way to an answer.

If we were to list all combinations of "black" and "white", that would be 1.4×10^{11} combinations. So this list likely is not what I meant by mentioning listing.

A shorter method is to list all combinations of two socks, or BB, BW, WW. The order does not matter in the end. In the BW case we do not have two socks of the same color, so the result cannot be two.

Now the combinations of three socks are BBB, BBW, BWW, WWW. In each of these we have two socks of the same color. So if we pull out three socks, we are guaranteed to have two matching socks. Since two did not work, the smallest number of socks you can pull without looking to have two of the same sock is **three**.

Another way to view the problem is by ticking off marks in the following for each sock drawn:

Black	White

Once you have a tick mark in both, the next mark must land in an occupied slot. So the longest sequence of marks without a duplicate is checking off one and then the other, or two marks. After two, you must mark an occupied slot.

This is a consequence of the *pigeonhole principle*. When there are more pigeons than holes, there must be at least two pigeons in one hole. You can use the same principle to prove that there must be two people in the Tri-cities area with the same number of hairs on their heads.

10.1.4 Problem 35: Listing

The simplest solution is to start listing numbers. The text's example of 28 shows you will not need to list more than 28 numbers. And because 1 has no factor other than itself, we can skip the first obvious entry.

$$\begin{aligned}
2 &\neq 1\\
3 &\neq 1\\
4 &\neq 1+2=3\\
5 &\neq 1\\
\mathbf{6} &= \mathbf{1}+\mathbf{2}+\mathbf{3}
\end{aligned}$$

We could construct the list without the prime numbers 2, 3, 5, *etc.* A prime number has only itself and 1 as factors, so we know that they cannot be perfect numbers.

Aside: What would the entry for 1 be? One has no factors other than itself. You would sum over $\emptyset$, the empty set. There is no single definition. In the context of this problem, we could define that sum to be zero. That would be consistent with the rest of the problem and perfectly ok. But it's not a completely standard definition.

Dealing with *vacuous* cases like the sum of entries of $\emptyset$ is tricky, but there is a general rule of thumb. Often the vacuous definition needs to be the *identity* of the operation. Here, zero is the identity element for addititon because $x + 0 = 0$ for all x. So it's a safe guess that the sum of the entries in the empty set could be defined to zero.

Another side: No one knows if there are infinitely many perfect numbers, or if there are any odd perfect numbers.

10.1.5 Problem 28: Following dependencies

Let M_1 be the initial amount, M_2 be the amount after buying the book, M_3 be the about after the train ticket, M_4 be the amount after lunch, then M_5 be the final about after the bazaar. Translating the problem into a sequence of equations,

$$\begin{aligned}
M_2 &= M_1 - 10,\\
M_3 &= M_2/2,\\
M_4 &= M_3 - 4,\\
M_5 &= M_4/2, \text{ and}\\
M_5 &= 8.
\end{aligned}$$

The only fully resolved data we have is $M_5 = 8$, so we start there. First $M_5 = 8 = M_4/2$, so $M_4 = 16$. Then $16 = M_3 - 4$ and $M_3 = 20$. Next, $20 = M_2/2$ so $M_2 = 40$. Finally, $40 = M_1 - 10$ and $\mathbf{M_1 = 50}$.

10.1.6 Problem 57: Following dependencies

This is an exercise in translating the words into mathematical relations and then working through the dependencies.

Translate each statement into an equation

$$\begin{aligned}
x_2 &= 3x_1, \\
x_3 &= x_2 + \frac{3}{4}x_2, \\
x_4 &= x_3/7, \\
x_5 &= x_4 - \frac{1}{3}x_4, \\
x_6 &= x_5^2, \\
x_7 &= x_6 - 52, \\
x_8 &= \sqrt{x_7}, \\
x_9 &= x_8 + 8, \\
x_{10} &= x_9/10, \text{ and} \\
x_{10} &= 2.
\end{aligned}$$

Following these backwards shows that

$$\begin{aligned}
x_9 &= 20, \\
x_8 &= 12, \\
x_7 &= 144, \\
x_6 &= 196, \\
x_5 &= 14, \\
x_4 &= 21, \\
x_3 &= 21 \cdot 7 (\text{note the next step divides by 7, so don't expand}), \\
x_2 &= 84, \text{and} \\
\mathbf{x_1} &= \mathbf{28}.
\end{aligned}$$

10.1.7 Problem 40: Bisection and guessing a range

See the write-up below.

10.1.8 Problem 52: Think about bisection

With eight coins, split them into two groups of four. One will be lighter. Split the lighter group of four into two groups of two. Again, one is lighter. Now the lighter group of two splits into two single coins. The lighter of the two coins is fake.

For the trick of two weighings, we first consider weighing two groups of *three* coins. If the groups are equal, we are left with the two remaining coins. Those two can be separated in one more weighing. If the groups of three are unequal, split the lighter one into three groups of one. Compare two of those single coins. If they are of the same weight, the left-over coin must be fake. Otherwise the lighter coin is the fake. Any path here requires only two weighings. This is an example *trisection*, separating the problem into three groups at each level.

10.1.9 Problem 56

A rotated square with each vertex located at the midpoint of the given square's sides will separate the kitties.

10.1.10 Problem 61: Look for a pattern

Calculating 1/7 to a few places shows $1/7 = 0.14285714285714\cdots$ The expansion appears to repeat in groups of six. Because $100 = 16 \cdot 6 + 4$, we expect that the 100^{th} digit after the decimal point is 8.

10.2 Making change

> How many ways can you make change for 60 cents using pennies, nickles, dimes, and quarters. Either take great care in forming a long list, or look for a relationship using smaller problems.
>
> A hint for a long list: Do you need to move pennies one at a time?
>
> A hint for a relationship: Consider the old example of 20 cents using pennies, nickles and dimes. How many ways are there to change 20 cents using only pennies and nickles? How many ways to change 20 cents minus one dime using all the coins? The relationship makes constructing a table much easier.

This is a classical problem used in discrete mathematics and introductory computer science classes, although often starting with a dollar to make tabling less practical.

There are 73 ways.

There are at most two quarters, at most six dimes, at most 12 nickles, and at most 60 pennies per line of a table with the following heading:

Pennies	Nickles	Dimes	Quarters	total

To generate the table, start with two quarters and then shift amounts over as in other problems. You use the conserved quantity, the total amount, to guide your next choice.

Another method for solving this problem is to set up recurrence relationships and build a slightly different and much, much shorter table.

Consider making change for an amount N. And consider four different ways for making such change:

A_N with only pennies,
B_N with nickles and pennies,
C_N with dimes, nickles, and pennies, and
D_N with quarters, dimes, nickes, and pennies.

Say we start at N and the full collection of possible coins. Then either the change contains a quarter or it does not. If it does contain with a quarter, then we change the remaining $N - 25$ in the same way, possibly with more quarters. If not, then we change N no quarters. So

$$D_N = C_N + D_{N-25}.$$

Similarly,

$$\begin{aligned} C_N &= B_N + C_{N-10}, \text{ and} \\ B_N &= A_N + B_{N-5}. \end{aligned}$$

We can begin constructing a table of values by N starting from the extreme case $N = 0$. There is only one way of making no change at all, so $A_0 = B_0 = C_0 = D_0 = 1$. There also is only one way of making change with pennies, so $A_N = 1$ for all N. And the relations above show we need only rows where N is a multiple of five and provide formulas for every entry.

The table is as follows:

N	A_N	B_N	C_N	D_N
0	1	1	1	1
5	1	2	2	2
10	1	3	4	4
15	1	4	6	6
20	1	5	9	9
25	1	6	12	13
30	1	7	16	18
35	1	8	20	24
40	1	9	25	31
45	1	10	30	39
50	1	11	36	49
55	1	12	42	60
60	1	13	49	$\mathbf{73 = 49 + 24}$

10.3 Writing out problems

10.3.1 Section 1.2, problem 9

Understanding the problem

We need to extend the sequence of interior regions to find the number of regions when there are seven and eight points. We have the first six entries of the sequence.

Devise a plan

Given the first six entries, we can form a successive difference table to extrapolate the sequence.

Carry out the plan

The table follows:

points	regions	$\Delta^{(1)}$	$\Delta^{(2)}$	$\Delta^{(3)}$	$\Delta^{(4)}$
1	1				
2	2	1			
3	4	2	1		
4	8	4	2	1	
5	16	8	4	2	1
6	31	15	7	3	1
7	$\boxed{\mathbf{57}}$	**26**	**11**	**4**	**1**
8	$\boxed{\mathbf{99}}$	**42**	**16**	**5**	**1**

Examine the solution

Computing $R(7) = \frac{1}{24}(7^4 - 6 \cdot 7^3 + 23 \cdot 7^2 - 18 \cdot 7 + 24) = 57$ and $R(8) = \frac{1}{24}(8^4 - 6 \cdot 8^3 + 23 \cdot 8^2 - 18 \cdot 8 + 24) = 99$ confirms the table's results. Also note that the table needed four columns to the right of the sequence to find the constant increment. This matches the degree, four, of the polynomial.

10.3.2 Section 1.2, problem 49

Understanding the problem

We need to inductively determine the formula for $N(n)$, the n^{th} nonagonal number. We have the following formulas:

$$\begin{aligned} H(n) &= \frac{n(4n-2)}{2}, \\ Hp(n) &= \frac{n(5n-3)}{2}, \text{ and} \\ O(n) &= \frac{n(6n-4)}{2}, \end{aligned}$$

for the hexagonal, heptagonal, and octagonal numbers.

Devise a plan

The plan is to look for a pattern in the formulas for $H(n)$, $Hp(n)$, and $O(n)$. That lets us predict $N(n)$.

Carry out the plan

All the formulas provided are of the form

$$\frac{n(an+b)}{2},$$

so we examine patterns in a and b.

The values of a are 4, 5, and 6. This suggests the next will be 7. And the values of b are -2, -3, and -4, suggesting to continue the pattern with -5. So we expect that the formula is

$$N(n) = \frac{n(7n-5)}{2}.$$

Examine the solution

To check, we verify that $N(6) = 111$. Indeed, $N(6) = 6 \cdot (42-5)/2 = 3 \cdot 37 = 111$, supporting the guess. While not a proof, this is a enough evidence to elevate the guess for $N(n)$ to a conjecture.

10.3.3 Section 1.3, problem 40

Understanding the problem

We are looking for a year. That year is 76 more than the birth year of one of the authors and is also a perfect square. The final answer is that year, x, minus 76.

Exploring the problem, we realize that the birth year in question must be within 76 years of publication of the text. The text was published in 2003. To stick to round numbers for estimation, we look for years between 1900 and 2100.

Devise a plan

Look for an integer between 1900 and 2100 that is a perfect square. Knowing that $\sqrt{1900} > 43$ and $\sqrt{2100} < 46$, we search the region $44 \leq i \leq 45$. Bisection here is overkill; there only are two choices after limiting the choices as above.

Carry out the plan

i	i^2	$i^2 - 76$
45	2025	1949
44	1936	1860

So the result is that Hornsby was born in **1949**, because 1860 is more than 76 years before 2003.

Examine the solution

Bisection here was unnecessary. Had we rounded outwards, though, and assumed only $43 \leq i \leq 46$, then bisection would have delivered the result in the first step.

10.4 Computing with numbers

10.4.1 Extra digits from 1/7

> Compute 1/7. Write down the number exactly as displayed. Then subtract what you have written from the calculator's or program's result. For a calculator, divide one by seven and then subtract off what you see without storing the result elsewhere. For a spreadsheet or other interface, divide one by seven. Then compute $1/7 - .14\cdots$ for whatever was displayed. What is the result? What did you expect? What result did others find?

With a computer using typical arithmetic, we may see 1/7 computed to be 0.142857142857143. Subtracting that off gives $-1.38777878078145 \times 10^{-16}$, not zero! Some computers (notably 32-bit Intel-like processors, although not under recent versions of Windows) may show a different result; they store intermediate results to extra precision.

With a calculator, you typically will see one or two non-zero digits. An 8-digit calculator might display 0.1428571. Subtracting that off may show any of 0, 4, 42, or 43 depending on how the calculator rounded and how many extra digits were kept. Most calculators keep a few extra digits past what is displayed.

10.4.2 Binary or decimal?

> Enter .1 into whatever device you use. Add .1 to it. Repeat eight more times, for a total of $10 \cdot .1$. Subtract 1. What is the result? What did you expect? What result did others find?

Using a *calculator*, you probably see zero, exactly what you expect from $10 \cdot .1 - 1$. Most hand-held calculators work with decimal arithmetic directly.

With most computers, you see $-1.11022302462516 \times 10^{-16}$. This is because .1 cannot be represented exactly in binary. The fraction 1/10 when converted to binary and computed does not terminate, just as 1/3 or 1/7 do not terminate in decimal.

Some systems run decimal arithmetic in software and also will produce zero. However, not all systems that show zero actually *have* zero stored as the result. Some spreadsheet software is guilty of "cosmetic rounding". They will display the result as zero but actually carry the binary version; what you see most certainly may not be what you get.

Chapter 11

Notes for reading graphs

Hopefully on 1 September.

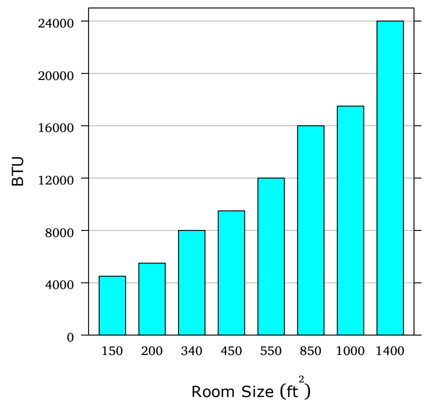
Notes also available as PDF. Images as slides are available PDF as well.

11.1 Reading graphs

Everyone knows the basics for most plots. Find the data you have and follow lines to find the result. But what about points in between?

Reading graphs is a form of *inductive* reasoning.

- Take care with your assumptions when reading graphs.
- Make your assumptions plain when creating graphs.
- Not all data need be graphed.
- Modern, measurable plots began roughly with William Playfair (1759-1823).

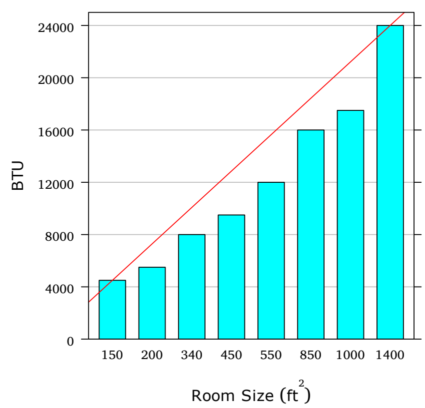


The text's Figure 10. So 850 square feet needs a 16 000 BTU air conditioner.

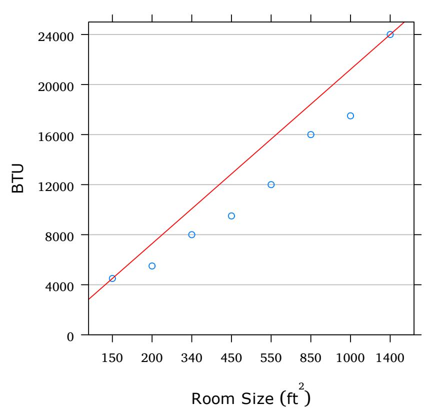
- What if you have a 900 square foot area? Might look between nearby points.
- Or a 100 square foot area? Only one nearby point.

Is there a relationship you can use?

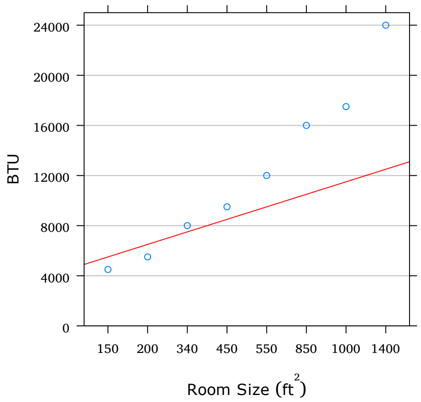
(Text figure's source: Carey, Morris, and James. *Home Improvement for Dummies*, IDG Books.)



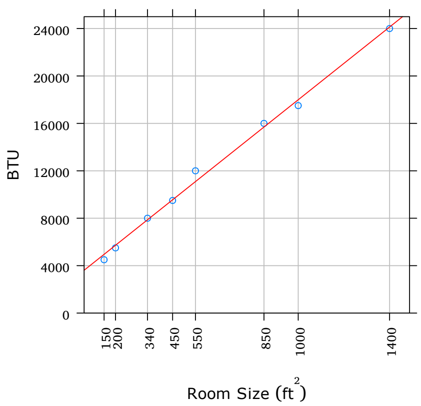
- A first thought is to use a line.
- People often build items to fit lines; it's how we tend to think.
- But the relationship here doesn't look like a line, does it?



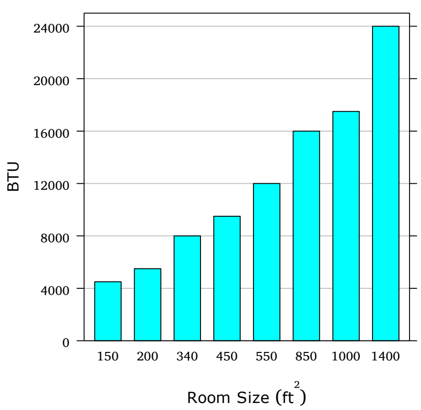
- As a first step, get rid of the bars.
  - The areas have no meaning.
  - But they aren't *bad* here, more on that later.
- We haven't changed the data, so this still does not appear to be a line.



- Trying a statistical line fit: Definitely no line here.
- *But look at the x axis.*
- Are the points spaced appropriately?



- Spread the points out, and suddenly we *do* have a line.
  - The BTU measurements likely are rounded, so not a perfect line.
- Now we can predict the BTUs needed for any size without having to poke at nearby points and estimate differences.



The points:

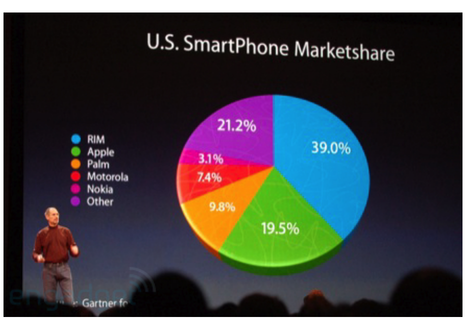
- Bar charts like these often are *tables* and not graphs.
- *Inductive reasoning*: Keep track of your assumptions when extrapolating visual relationships.

- Remember in the bar chart: Areas did not matter.
- People are very, very bad at judging areas.
- Given the one baloon represents 12%, how large is the one next to it?

(From the Onion (http://www.theonion.com))

- Given the one baloon represents 12%, how large is the one next to it?
- **22%**
- This is the Onion, but the graph *is* to scale.
- Not by area, but by *length*.
- But you see area first...

(From the Onion (http://www.theonion.com))



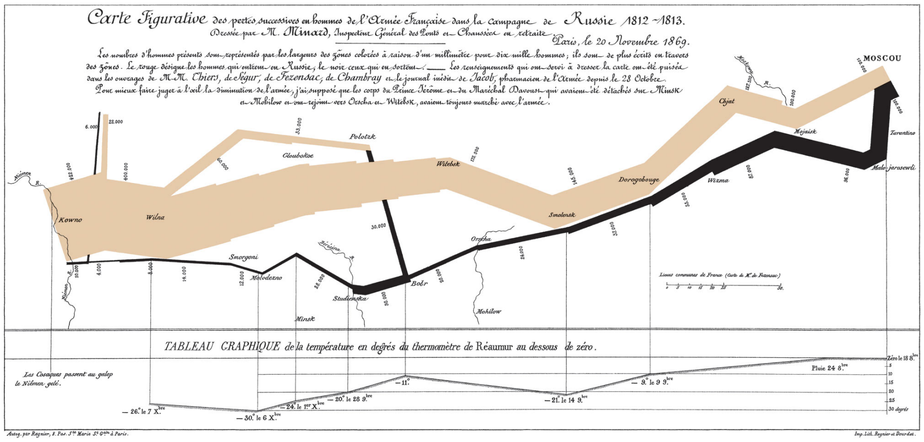
- Also beware graphs with too much of a "slant".
  - (multiple meanings here)
- In a "pie chart", areas are the data.
- But people are very, very bad at judging areas.
- Which is larger, 19.5% or 21.2%?
- Who is the 19.5% in this image?
- **Avoid 3D effects!**

(at Macworld 2008, photo from Ryan Block of Engadget (http://www.engardget.com))

Vendor	US market share (%)
RIM	39.0
Apple	19.5
Palm	9.8
Motorola	7.4
Nokia	3.1
other	21.2

- Never be afraid of using small tables.
- Is there a problem with *other* being the second largest?
- *other*: LG, Samsung, Ericsson, . . .
- Knowing your premises:
  - US-only.
    - Nokia is #1 world-wide (40%)
    - Samsung is #2 (15%)
    - Motorola is #3 (10%)
  - No history in this table, no idea about future.

This does not imply area is useless.

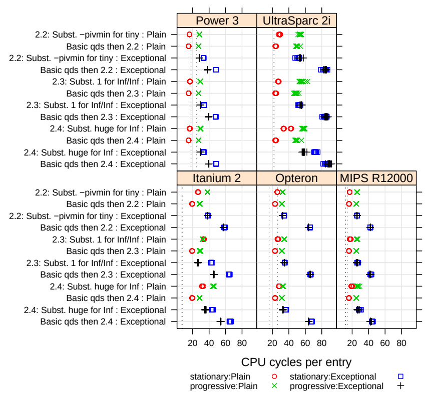


- Graphic by Charles Joseph Minard in 1869.
- Title: *Carte figurative des pertes successives en hommes de l'Armée Française dans la campagne de Russie 1812-1813*
- Depicts Napolean's 1812 march on Moscow and subsequent disaster.
- Displays many variables in one image:
  - location on the map,
  - direction by color,
  - size by width, and
  - temperature during the retreat by the graph on the bottom.
- At the time, an anti-war graphic!
- Considered one of the best graphical displays of data across all of history.

11.2 Creating a graphical depiction of data

- Begin with questions:
  - What should the reader take away?
  - What does the data really imply?
  - One graph should not have too many messages.
- Help the reader form correct comparisons. Items to compare should
  - be represented similarly,
  - and lie close together.
- People judge lengths much better than areas.
- Show causality, and avoid inferring causality where none exists.
  - Plot unrelated quantites on different graphs, not on opposite axes of the same graph.
- Use numbers and words.
- Do not use visual effects unless they directly portray data.
  - Extraneous symbols were coined “chartjunk” by Edward Tufte.
  - Many affects can distort data, particularly 3-d affects.

Will walk through some of my thoughts while creating graphics for a highly technical paper.

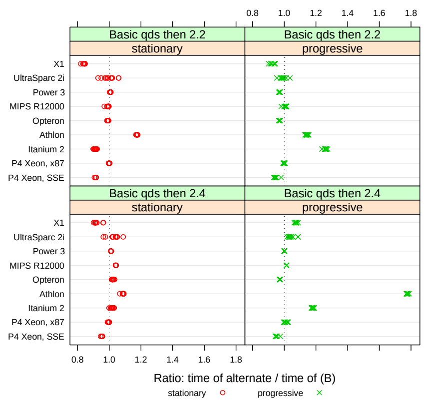


- Purpose: Display all our experimental data without much interpretation.
- Too much data for a simple table?
- On the left: algorithms and data cases (plain v. exceptional)
- Blocks: Specific processors
- Below: CPU/processor cycles per array entry
- Dotted vert. line: CPU cycles for a critical operation
- Colors and symbols: "direction" of algorithm and data cases (*repeated!*)
- *Graph allows simple comparisons of our raw data.*

(from Marques, Riedy, and Vömel. *Benefits of IEEE-754 features in modern symmetric tridiagonal eigensolvers*)

- Purpose: Determine if CPUs impose penalties on certain arithmetic features.
- Each algorithm (green bars) uses a different feature.
- Ratio of "careful" over "plain" shows a slow-down.
- Find outliers by looking down and across:
  - One direction ("progressive") encounters more problems than the other ("stationary")?
  - Missing data here: "stationary" ran far more slowly, slow-down hidden by total cost.

(from Marques, Riedy, and Vömel. *Benefits of IEEE-754 features in modern symmetric tridiagonal eigensolvers*)



- Purpose: Begin interpreting the data to determine which *single* algorithm should be used.
- Chose one algorithm, "B", to normalize others (2.2, 2.4).
  - Reviewers (and authors) missed a typo. "B" should be 2.3.
- Could this have been a table?
  - More than a page of data in tabular form.
  - Can be summarized by statistics (median and percentiles).
  - Summaries were in the text.
- Does this serve its purpose?
  - With hindsight, not really.
  - Summary in the text was better.
  - This plot was unnecessary.

(from Marques, Riedy, and Vömel. *Benefits of IEEE-754 features in modern symmetric tridiagonal eigensolvers*)

11.3 Graph galleries and resources

- Gallery of Data Visualization; The Best and Worst of Statistical Graphics: `http://www.math.yorku.ca/SCS/Gallery/`
- Edward Tufte's site: `http://www.edwardtufte.com`
- Example graphs, some good, some not so good: `http://addictedtor.free.fr/graphiques/`
- Other examples or essays:
  - `http://www.dmreview.com/issues/20050101/1016296-1.html`
  - `http://www.bella-consults.com/square-pies`

Chapter 12

Homework for reading graphs

12.1 Homework

Practice is absolutely critical in this class.

Groups are fine, turn in your own work. Homework is due in or before class on Mondays.

- Section 1.4, problem 54. And critique the graph using the number of subscribers (in 1999) for C-Band and Primestar.

Note that you *may* email homework. However, I don't use Microsoft$^{\text{TM}}$ products (*e.g.* Word), and software packages are notoriously finicky about translating mathematics.

If you're typing it (which I advise just for practice in whatever tools you use), you likely want to turn in a printout. If you do want to email your submission, please produce a PDF or PostScript document.

Chapter 13

Notes for the third week: set theory

Notes also available as PDF.

13.1 Language of set theory

- We will cover just enough set theory to use later.
- Cardinalities are important for probability. We don't have time to cover probability sufficiently well, so we will not explore the sizes of sets deeply.
- This is known as naïve set theory. We do not define absolutely everything, nor do we push set theory's logical limits. Much.

Goals:

- Impart some of the language necessary for later chapters.
- Practice reasoning in a formal setting.
    - One key aspect is what to do in extreme cases like empty sets.
- Set up straight-forward examples for logic.

13.2 Basic definitions

To start, we require unambiguous definitions of terms and items. When a term or item is unambiguously defined, it is called *well-defined*.

set An *unordered* collection of *unique* elements.

- Curly braces: $\{A, B, C\}$ is a set of three elements, A, B, and C.
- Order does not matter: $\{\text{cat}, \text{dog}\}$ is the same set as $\{\text{dog}, \text{cat}\}$.
- Repeated elements do not matter: $\{1, 1, 1\}$ is the same set as $\{1\}$.
- Can be *implicit*: $\{x \mid x \text{ is an integer}, x > 0, x < 3\}$ is the same set as $\{1, 2\}$.
- Read the implicit form as "the set of elements x *such that* x is an integer, $x > 0$, and $x < 3$". Or "the set of elements x *where* ..."
- Other symbols that sometimes stand for "such that": :, $\ni$ (reversed $\in$)
- Implicit (or set-builder) form can include formula or other bits left of the bar. $\{3x \mid x \text{ is a positive integer}\}$ is the set $\{3, 6, 9, \ldots\}$.

element Any item in a set, even other sets. (Also entry, member, item, *etc.*)

- This is not ambiguous. If something is in a set, it is an item of that set. It doesn't matter if the item is a number or a grape.
- $\{A, \{B, C\}\}$ is a set of *two* elements, A and $\{B, C\}$.
- None of the following are the same: $\{A, \{B, C\}\}$, $\{A, B, C\}$, $\{\{A, B\}, C\}$.

empty set Or null set. Denoted by $\emptyset$ rather than $\{\}$.

- This is a *set* on its own.
- $\{\emptyset\}$ is the set of the empty set, which is not empty.
- Think of sets as bags. An empty bag still is a bag, and if a bag contains an empty bag, the outer bag is not empty.
- Implicit definitions can hide empty sets.
- For example, the set $\{x \mid x \text{ is an odd integer divisible by } 2\}$ is $\emptyset$.

singleton A set with only one element.

- $\{1\}$ and $\{\emptyset\}$ both are singletons (or sometimes singleton sets).

13.3 Translating sets into (and from) English

From English:

- The days of the week:
  - $\{\text{Monday}, \text{Tuesday}, \text{Wednesday}, \text{Thursday}, \text{Friday}, \text{Saturday}, \text{Sunday}\}$

  - Of course, we're using a *representation* of the days and not the days themselves. That is how we reason about things; we model them and represent them by symbols.
- The days when homework is due:
  - $\{25^{\text{th}}$ of August, 1^{st} of September, $\ldots\}$
  - We *could* list them all.
  - { every Monday after the 18^{th} of August 2008 until after the 1^{st} of December }
  - Or: $\{x \mid x$ is a Monday, x is after the 18^{th} of August, and x is on or before the 1^{st} of December $\}$

To English:

- $\{2, 3, 4\}$:
  - The set containing two, three, and four.
- $\{x \mid x$ is an integer and $x > 0\}$:
  - The positive integers, also called the counting numbers or the natural numbers.
  - Often written as $\mathbb{J}^+$. The integers often are written as $\mathbb{J}$ (because the "I" form can be difficult to read), rationals as $\mathbb{Q}$ (for quotients), the reals as $\mathbb{R}$.
- $\{2x - 1 \mid x \in \mathbb{J}^+\}$
  - The set whose members have the form $2x - 1$ where x is a positive integer.
  - Cannot list all the entries; this is an *infinite* set.
  - Here, the odd integers.

13.4 Relations

element of The expression $x \in A$ states that x is an element of A. If $x \notin A$, then x is *not* an element of A.

- $4 \in \{2, 4, 6\}$, and $4 \notin \{x \mid x$ is an odd integer $\}$.
- There is no x such that $x \in \emptyset$, so $\{x \mid x \in \emptyset\}$ is a long way of writing $\emptyset$.

subset If all entries of set A also are in set B, A is a subset of B.

superset The reverse of subset. If all entries of set B also are in set A, then A is a superset of B.

proper subset If all entries of set A also are in set B, but some entries of B are *not* in A, then A is a *proper* subset of B.

- $\{2, 3\}$ is a proper subset of $\{1, 2, 3, 4\}$.

equality Set A equals set B when A is a subset of B and B is a subset of A.

- Order does not matter. $\{1, 2, 3\} = \{3, 2, 1\}$.

The symbols for these relations are subject to a little disagreement.

- Many basic textbooks write the subset relation as $\subseteq$, so $A \subseteq B$ when A is a subset of B. The same textbooks reserve $\subset$ for the *proper* subset. Supersets are $\supset$.
- This keeps a superficial similarity to the numerical relations $\leq$ and $<$. In the former the compared quantities may be equal, while in the latter they must be different.
- Most mathematicians now use $\subset$ for any subset. If a property requires a "proper subset", it often is worth noting specifically. And the only non-"proper subset" of a set is the set itself.
- Extra relations are given for emphasis, *e.g.* $\subsetneq$ or $\subsetneqq$ for proper subsets and $\subseteq$ or $\subseteqq$ to emphasize the possibility of equality.
- Often a proper subset is written out: $A \subset B$ and $A \neq B$.
- **I'll never remember to stick with the textbook's notation. My use of $\subset$ is for subsets and not proper subsets.**

13.5 Translating relations into (and from) English

From English:

- The train has a caboose.
  - It's reasonable to think of a train as a set of cars (they can be reordered).
  - The cars are the members.
  - Hence, caboose $\in$ train
- The VI volleyball team consists of VI students.
  - VI volleyball team $\subset$ VI students

- There are no pink elephants.
  - pink elephants = $\emptyset$

To English:

- $x \in$ today's homework set.
  - x is a problem in today's homework set.
- Today's homework $\subset$ this week's homework.
  - Today's homework is a subset of this week's homework.

13.6 Consequences of the set relation definitions

Every set is a subset of itself. Expected.

If $A = B$, then every member of A is a member of B, and every member of B is a member of A. This is what we expect from equality, but we did not define set equality this way. Follow the rules:

- $A = B$ imples $A \subset B$ and $B \subset A$.
- Because $A \subset B$, every member of A is a member of B.
- Because $B \subset A$, every member of B is a member of A.

The empty set $\emptyset$ is a subset of all sets. Unexpected! This is a case of carrying the formal logic to its only consistent end.

- For some set A, $\emptyset \subset A$ if every member of $\emptyset$ is in A.
- But $\emptyset$ has no members.
- Thus all of $\emptyset$'s members also are in A.
- This is called a *vacuous* truth.

The alternatives would not be consistent, but proving that requires more machinery that we need.

13.7 Visualizing two or three sets: Venn diagrams

Also known as *Venn diagrams*.

yes, at some point I will draw some and stick them in the notes.

13.8 Operations

union The *union* of two sets A and B, denoted by $A \cup B$, is the set consisting of all elements from A and B.

- $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
- Remember repeated elements do not matter: $\{1,2\} \cup \{2,3\} = \{1,2,3\}$.

intersection The *intersection* of two sets A and B, denoted $A \cap B$, is the set consisting of all elements that are in *both* A and B.

- $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.
- $\{1,2\} \cap \{2,3\} = \{2\}$.
- $\{1,2\} \cap \{3,4\} = \{\} = \emptyset$.

set difference The *set difference* of two sets A and B, written $A \setminus B$, is the set of entries of A that are not entries of B.

- $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$.
- Sometimes written as $A - B$, but that often becomes confusing.

If A and B share no entries, they are called *disjoint*. One surprising consequence is that every set A has a subset disjoint to the set A itself.

- No sets (not even $\emptyset$) can share elements with $\emptyset$ because $\emptyset$ has no elements.
- So all sets are disjoint with $\emptyset$.
- The empty set $\emptyset$ is a subset of all sets.
- So all sets are disjoint with at least one of their subsets!

Can any other subset be disjoint with its superset? *No.*

13.8.1 Similarities to arithmetic

Properties of arithmetic:

commutative $a + b = b + a$, $a \cdot b = b \cdot a$

associative $a + (b + c) = (a + b) + c$, $a(bc) = (ab)c$

distributive $a(b + c) = ab + ac$

Which of these apply to set operations *union* and *intersection*? (Informally. Formally we must rely on the properties of *and* and *or*.)

If $C = A \cup B$, then $C = \{x \mid x \in A \text{ or } x \in B\}$. Reversing the sets does not matter, so $C = A \cup B = B \cup A$. The union is **commutative**. Similary, if $D = A \cup (B \cup C)$, we can write D in an implicit form and see that $D = (A \cup B) \cup C$ to see that the union is **associative**.

The same arguments show that set intersection is **commutative** and **associative**.

For the distributive property, which is similar to addition and which to multiplication? A gut feeling is that unions *add*, so try it.

$$\begin{aligned}
A \cap (B \cup C) &= \{x \,|\, x \in A \text{ and } x \in B \cup C\} \\
&= \{x \,|\, x \in A \text{ and } (x \in B \text{ or } x \in C)\} \\
&= \{x \,|\, (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)\} \\
&= (A \cap B) \cup (A \cap C)
\end{aligned}$$

But with sets, *both* operations distribute:

$$\begin{aligned}
A \cup (B \cap C) &= \{x \,|\, x \in A \text{ or } x \in B \cap C\} \\
&= \{x \,|\, x \in A \text{ or } (x \in B \text{ and } x \in C)\} \\
&= \{x \,|\, (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)\} \\
&= (A \cup B) \cap (A \cup C)
\end{aligned}$$

The rules of set theory are intimately tied to logic. Logical operations dictate how set operations behave. We will cover the properties of logic in the next chapter.

13.9 Translating operations into English

To English:

- $(A \cup B) \cap C$
  - The set consisting of members that are in C and either of A or B.
- $(A \cap B) \cup C$
  - The set consisting of members that are in C or in both of A or B.

13.10 Special operations

The complement and cross-product operations require extra definitions.

13.10.1 Universes and complements

universe A master set containing all the other sets in the current context.

complement The *complement* of a set A is the set of all elements in a specified universal set U that are *not* in A.

- $A^c = \{x \mid x \notin A \text{ and } x \in U\} = U - A$.
- Sometimes written as A' or $\bar{A}$.

- It's not always necessary to define a universal set.
- And there is no "universal" universal set.
- Because $A^c = U \setminus A$, many people avoid the complement completely.
- The complement is useful to avoid writing many repeated $U \setminus A$ operations that share the same universal set.

13.10.2 Tuples and cross products

tuple An *ordered* collection of elements, (A, B, C).

- When only two elements, this is an *ordered pair*.
- Think of coordinates in a graph, (x, y).
- So $(x, y) \neq (y, x)$ in general (*i.e.* when $x \neq y$).

cross product A *set* of all *ordered pairs* whose entries are drawn from two sets.

- $A \times B = \{(x, y) \mid x \in A, y \in B\}$.

Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$.

Then $A \times B = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)\}$ and $B \times A = \{(b_1, a_1), (b_1, a_2), (b_2, a_1), (b_2, a_2)\}$. Because $(a_1, b_1) \neq (b_1, a_1)$ in general, $A \times B \neq B \times A$ in general.

When does $A \times B = B \times A$?

13.11 Cardinality and the power set

cardinality The *cardinality* of a set A is the number of elements in A. Often written as $|A|$. The text uses $n(A)$.

- If $A = \{1, 2, 3\}$, then $|A| = 3$.
- What is $|\emptyset|$? 0.

power set The *power set* of a set A is the set of all subsets of A.

- Often denoted as $\mathcal{P}(A)$, but this is used rarely enough that the notation always needs defined.

What is the cardinality of the power set of A?

- What is cardinality of the power set of $\emptyset$?
  - All sets are subsets of themselves, and the empty set is a subset of itself.
  - Then $\mathcal{P}(\emptyset) = \{\emptyset\}$, and $|\mathcal{P}| = |\{\emptyset\}| = 1$.
- What is the powerset of a set with one element, let's say $\{1\}$?
  - There are two subsets, $\emptyset$ and the set itself $\{1\}$.
  - $\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$, and $|\mathcal{P}(\{1\})| = 2$.
- Two elements, say $\{1, 2\}$?
  - $\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.
  - $|\mathcal{P}(\{1, 2\})| = 4$.
- So the powerset with zero entries has size 1, one entry has size 2, two has size 4, ...

What is the cardinality of $A \cup B$?

- Sets do not contain repeated members, so the union cannot be simply the sum of its arguments.
- The intersection contains one copy of all the shared members.
- So to count every item *once* the cardinality of the union is the sum of the cardinalities of the sets minus the cardinality of the intersection.
- $|A \cup B| = |A| + |B| - |A \cap B|$.
- Known as the *inclusion-exclusion principle*.
- Extends to more sets, but you must be careful about counting entries once!
  - $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$.

Chapter 14

Homework for the third week: set theory

14.1 Homework

Practice is absolutely critical in this class.

Groups are fine, turn in your own work. Homework is due in or before class on Mondays.

Most of these problems are purely mechanical. This is less work than it appears.

- Section 2.1:
  - Problems 1-8
  - Problems 11 and 17
  - Problems 30 and 32
  - Problems 62, 63, and 66
  - Problems 68, 71, 74, and 78
  - Problem 92
- Section 2.2:
  - Problems 8, 10, 12, 14
  - Even problems 24-34, using the *text*'s definitions of subset and proper subset
- Section 2.3:

- Problems 1-6
- Problems 10, 17, 18, 23, 24
- Problem 31
- Problem 33, rephrase using complements with respect to the common "universal" set $A \cup B \cup C$.
- Problems 61, 62
- Problems 72, 73
- Problems 117, 118, 121-124

Note that you *may* email homework. However, I don't use Microsoft$^{\mathrm{TM}}$ products (*e.g.* Word), and software packages are notoriously finicky about translating mathematics.

If you're typing it (which I advise just for practice in whatever tools you use), you likely want to turn in a printout. If you do want to email your submission, please produce a PDF or PostScript document.

Chapter 15

Solutions for third week's assignments

Also available as PDF.

15.1 Section 1.4, problem 54

According to the percentages, Primestar has 16% of 12 million or 1.92 million. C-Band then has 15% or 1.8 million. Primestar has 120 thousand more subscribers.

However, the slices appear of drastically different sizes. I suspect the satellite dish is tilted "upwards" like a real dish, distorting the slices' areas.

15.2 Section 2.1

15.2.1 Problems 1-8

1. C
2. G
3. E
4. A
5. None of the above! They meant B, but $1 = 2^0$ is a positive integer and a power of two. The authors meant "two raised to the power of each of the five least positive integers". *I hadn't realized this at first, or else I would not have given this one.*

6. D
7. H
8. F

15.2.2 Problems 11 and 17

11. $\{0, 1, 2, 3, 4\}$
17. $\{2, 4, 8, 16, 32, 64, 128, 256\}$

15.2.3 Problems 30 and 32

30. $\{x|x \text{ is an even natural number}\}$ is a direct translation, but $\{2x|x \in \mathbb{J}^+\}$ is shorter. Another possibility is $\{x|x > 0, x \text{ is an even integer}\}$.
32. One form is $\{35 + 5i|i \in \mathbb{J}, 0 \leq i \leq 12\}$.

15.2.4 Problems 62, 63, and 66

62. $-12 \notin \{3, 8, 12, 18\}$.
63. $0 \in \{-2, 0, 5, 9\}$.
66. $\{6\} \notin \{3, 4, 5, 6, 7\}$. But note that $\{6\} \subset \{3, 4, 5, 6, 7\}$.

15.2.5 Problems 68, 71, 74, and 78

68. false
71. true
74. true
78. true (assuming a typical meaning for "...")

15.2.6 Problem 92

Part a. Three chocolate bars are contain a total of 660 calories. The point of this exercise is to ensure you recognize that sets are unordered, so $\{r, s\} = \{s, r\}$ and you only include it once. The list is as follows: $\{r\}$, $\{r, s\}$, $\{r, c\}$, $\{r, g\}$, $\{r, v\}$, $\{s, c\}$, and $\{s, g\}$.

Part b. Five bars is 1100 calories. The list is $\{r, s, v\}$, $\{r, s, g\}$, $\{r, s, c\}$, $\{r, c, v\}$, $\{r, c, g\}$, and $\{r, g, v\}$.

15.3 Section 2.2

15.3.1 Problems 8, 10, 12, 14

8. $\{M, W, F\} \not\subset \{S, M, T, W, Th\}$.
10. $\{\mathrm{a}, \mathrm{n}, \mathrm{d}\} \subset \{\mathrm{r}, \mathrm{a}, \mathrm{n}, \mathrm{d}, \mathrm{y}\}$.
12. $\emptyset \subset \emptyset$.
14. $\{2, 1/3, 5/9\} \subset \mathbb{Q}$.

15.3.2 Even problems 24-34

24. true
26. false
28. false
30. true
32. false
34. false

15.4 Section 2.3

15.4.1 Problems 1-6

1. B
2. F
3. A
4. C
5. E
6. D

15.4.2 Problems 10, 17, 18, 23, 24

10. $Y \cap Z = \{b, c\}$.
17. $X \cup (Y \cap Z) = \{a, b, c, e, g\}$.
18. $Y \cap (X \cup Z) = \{a, b, c\} = Y$ because $Y \subset X \cup Z$.
23. $X \setminus Y = \{e, g\}$.
24. $Y \setminus X = \{b\}$.

15.4.3 Problem 31

The set consisting of all the elements of A along with those elements of C that are not in B.

15.4.4 Problem 33

The set consisting of elements in A but not in C as well as elements in B but not in C. If we consider the union of A, B, and C to be the universal set, then this is the set of all elements in A complement along with all elements in B complement.

15.4.5 Problems 61, 62

61. $X \cup \emptyset = X$, and the conjecture is that the union of any set with the empty set is the set itself.
62. $X \cap \emptyset = \emptyset$, and the conjecture is that the intersection of any set with the empty set is the empty set.

15.4.6 Problems 72, 73

72. $A \times B = \{(3,6),(3,8),(6,6),(6,8),(9,6),(9,8),(12,6),(12,8)\}$
$B \times A = \{(6,3),(8,3),(6,6),(8,6),(6,9),(8,9),(6,12),(8,12)\}$
73. $A \times B = \{(d,p),(d,i),(d,g),(o,p),(o,i),(o,g),(g,p),(g,i),(g,g)\}$.
$B \times A = \{(p,d),(i,d),(g,d),(p,o),(i,o),(g,o),(p,g),(i,g),(g,g)\}$, alas, no pigdog in sight.

15.4.7 Problems 117, 118, 121-124

117. $A \setminus B = A$ implies that $A \cap B = \emptyset$.
118. $B \setminus A = A$ is true only when $B = A = \emptyset$.
121. If $A \cup \emptyset = \emptyset$, then $A = \emptyset$.
122. $A \cap \emptyset = \emptyset$ for any set A.
123. If $A \cap \emptyset = A$, then $A = \emptyset$.
124. $A \cup \emptyset = A$ for all sets A.

Part III

Notes for chapters 4, 5, and 6

Chapter 16

Notes for the fourth week: symbolic logic

Notes also available as PDF.

16.1 Language of logic

Goals:

- Determine when statements are **logical statements**.
- Recognize and transform between equivalent logical statements.
- Negate logical statements and quantified logical statements correctly.
- Apply the logical rules of deduction and follow *if-then* chains formally.

We start with definitions:

logical statement A declaration that is either true or false but not both. For example:

> Today is Monday.

Languages contain many statements and declarations that are not logical statements:

> Symbolic logic is fun.

While that is a declaration, it is neither true nor false *in general*. An example of a logical statement from set theory,

> $x \in A$.

negation A logical statement over the same topics that is false if the original statement is true or true if the original is false. The statement

> My dog has fleas.

has as its negation

> My dog does not have fleas.

However, a statement about my cat(s) cannot be a negation of either of the above regardless of which statements are true or false.

quantifier When a statement applies to *all*, *some*, *every* of something, the statement is **quantified**. The word denoting how many is the **quantifier**. This is where negation becomes tricky. For example, the statement

> All dogs have fleas.

has as its negation

> Some dogs do not have fleas.

Its negation is **not**

> All dogs do not have fleas.

16.2 Symbolic logic

Expressing logical statements with symbols lets us focus on manipulating the logic itself. We can talk about the dog having fleas without mentioning dogs or fleas.

Variables like p and q can take the values true or false. Like many items, true and false are given different symbols by different authors. Common symbols include

true	false
T	F
1	0
$\top$	$\perp$

I will use 1 and 0.

16.3 Logical operators and truth tables

The first operator is **negation**. This is a **unary** operation; it applies to a single operand. Two symbols are commonly used for the negation operator, $\neg$ and $\sim$, as is adding a bar to a variable, $\overline{p}$. I will stick with the symbol $\neg$ for most cases.

With one operand, listing all possible inputs and outputs is simple. The list is also called a **truth table**. The truth table for $\neg$ is as follows:

p	$\neg p$
1	0
0	1

Programming languages may represent $\neg$ with operators or functions like `!`, `not()`, `.NOT.`, `not`.

When **and** joins pieces of a logical statement, we understand that the statement as a whole it true only if both pieces are true. This is the **conjunction** operator. The *conjunction* operator $\wedge$ is the symbol we use to represent the same idea. The truth table for the $\wedge$ operator has four lines and is as follows:

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

The *and* operator sometimes is written as multiplication, or just pq. Most often, that is paired with using a bar for negation, so $p \wedge \neg q$ is written $p\overline{q}$. This is common notation in electrical engineering. Occasionally the negation will be written as q'. Programming languages may represent $\wedge$ with operators or functions like `&&`, `and()`, `.AND.`, `and`.

The English word **or**, however, has quite a few different meanings. Sometimes we mean the logical *exclusive-or*, where one choice rules out the other, and sometimes we mean logical *or*, where both choices are possible.

In symbolic logic, the **disjunction** operator, $\vee$, is the latter type of *or*. The *disjunction* is true when either sub-clause is true:

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

In electrical engineering, *or* often is represented by $+$, so $p \vee \neg q$ would be written $p + \overline{q}$. Programming languages may represent $\vee$ with operators or functions like `||`, `or()`, `.OR.`, `or`.

The **exclusive-or** operator, which we will denote $\oplus$, is true whenever exactly one operand is true:

p	q	$p \oplus q$
1	1	0
1	0	1
0	1	1
0	0	0

Mathematically, the symbol for the *exclusive-or* operator is not particularly standardized. The symbol $\veebar$ sometimes appears, as does the operator xor. We will expand on the notation $\oplus$ once we discuss addition of binary numbers.

Note that $\oplus$ often is not considered a core operator. We can write $p \oplus q$ as $(p \vee q) \neg (p \wedge q)$.

Programming languages rarely provide $\oplus$ logical operators, although they often provide *exclusive-or* operators on the binary representations of integers. More on those in the next chapter. But if you notice, this is the negation of equality. So programming languages express the logical *exclusive-or* by negating the equality of two logical expressions.

In symbolic logic, equality of two logical statements is **equivalence**. When expressed as an operator, equivalence takes the symbols $\equiv$, $\Leftrightarrow$, or $\leftrightarrow$. The reason for the arrow forms will become clear soon.

p	q	$p \equiv q$
1	1	1
1	0	0
0	1	0
0	0	1

Again, $\equiv$ is not a core operator. $p \equiv q$ is the same statement as $(p \wedge q) \vee (\neg p \wedge \neg q)$.

Programming languages represent equality with `==`, `=`, `equal?`, and many other forms.

A logical statement that always is true is a **tautology**. So $p \vee \neg p$ is a tautology:

> The sky is periwinkle or the sky is not periwinkle.

Symbolically,

$$\models p \vee \neg p.$$

A statement that always is false is a **contradiction**. Here $p \wedge \neg p$ is an example:

> The sky is blue and the sky is not blue.

A statement that is neither always true nor always false is logically **contingent**. So pq is contingent on the values of p and q.

We use the symbol for tautology, $\models$, to make certain equivalence statements unambiguous. Because we defined $\equiv$ as an operator above, the plain statement

$$p \vee q \equiv q \vee p$$

appears contingent on its values. By adding $\models$, we make it clear that we are asserting that the statement is always true. To state that $p \vee q$ is the same as $q \vee p$, we say

$$\models p \vee q \equiv q \vee p.$$

16.4 Properties of logical operators

Three properties from arithmetic also hold for the core logical operators $\wedge$ and $\vee$:

commutative In language, *and* and *or* are commutative, or p and q is the same as q and p. Symbolically, $\models p \wedge q \equiv q \wedge p$ and $\models p \vee q \equiv q \vee p$.

associative Again, linguistically we don't use parenthesis. Run-on statements are just as true or false as well-structured ones, so we expect *and* and *or* to be associative. We could build a large truth table to verify this, but for now let us just state that $\models (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ and $\models (p \vee q) \vee r \equiv p \vee (q \vee r)$.

distributive As with sets, both operations distribute over the other. So $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ and $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

As a demonstration of using a truth table to show equivalence,

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Which of these hold for $\oplus$ and $\equiv$? Over which operations may each be distributed?

16.5 Truth tables and logical expressions

16.5.1 De Morgan's laws

As an example of truth tables and a demonstration of some very useful logical laws, we examine De Morgan's laws.

In arithmetic, we know that $-(1 + 2) = -1 + -2$. Logical operations are similar to arithmetic, but not *that* similar.

We could reason about the two rules, but they serve as succinct examples of truth tables.

- $\neg(p \vee q) \equiv \neg p \wedge \neg q$:

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$p \vee q$	$\neg(p \vee q)$
1	1	0	0	0	1	0
1	0	0	1	1	0	1
0	1	1	0	1	0	1
0	0	1	1	1	0	1

- $\neg(p \wedge q) \equiv \neg p \vee \neg q$:

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \wedge q$	$\neg(p \wedge q)$
1	1	0	0	0	1	0
1	0	0	1	0	1	0
0	1	1	0	0	1	0
0	0	1	1	1	0	1

16.5.2 Logical expressions from truth tables

Say we are provided with a truth table showing all possible values for an unknown logical statement $f(p, q)$. From that truth table, we can construct a logical statement equivalent to $f(p, q)$.

Take:

p	q	$f(p,q)$
1	1	0
1	0	1
0	1	0
0	0	1

To construct f, we can combine all the true outputs with $\vee$. The terms to be combined are the $\wedge$ of all the variables. In each $\wedge$ expression, each variable is negated to make its value true.

So in the above table, there are two true entries. One has $p \equiv 1$ and $q \equiv 0$, while the other has $p \equiv 0$ and $q \equiv 0$. The two corresponding $\wedge$-statements are $p \wedge \neg q$ and $\neg p \wedge \neg q$. So an equivalent statement is

$$\models f(p, q) \equiv (p \wedge \neg q) \vee (\neg p \wedge \neg q).$$

Alternately, we can list the false entries and combine their negations. This provides another equivalent statement,

$$\models f(p, q) \equiv \neg\left((p \wedge q) \vee (\neg p \wedge \neg q)\right).$$

We can use De Morgan's laws to show these are the same:

$$\begin{aligned} \models \neg((p \wedge q) \vee (\neg p \wedge \neg q)) &\equiv \neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q) \\ &\equiv (\neg p \vee \neg q) \wedge (p \vee q) \end{aligned}$$

Neither of these are the *simplest* possible expression, but each is equivalent to $f(p, q)$ and is constructed by a straight-forward recipe.

To simplify this expression, we use the distributive property and the fact that $\models p \vee \neg p$,

$$\begin{aligned} \models f(p, q) &\equiv (p \wedge \neg q) \vee (\neg p \wedge \neg q) \\ &\equiv (p \vee \neg p) \wedge \neg q \\ &\equiv 1 \wedge \neg q \\ &\equiv \neg q. \end{aligned}$$

There is another method for simplifying expressions which we will not cover. If you are interested in a more visual method for simplifying expressions over a few variables, look up Veitch charts or Karnaugh maps from electrical engineering. With those forms, you draw a 2-d truth table and cover the true values with boxes of area 2^k. This mechanism is particularly useful when you have *don't care* values, places where the output value does not matter.

16.6 Conditionals

16.6.1 English and the operator $\rightarrow$

One (compound) operator we have not mentioned so far is the **conditional**, the symbolic form of an *if-then* rule. A statement like

> If the sky is blue, then it is not raining.

is translated to

the sky is blue $\rightarrow$ it is not raining,

or $p \rightarrow q$ using only symbols. Here p is the **antecedent** and q is the **consequent**.

Many English statements are forms of conditionals. For example,

- It is not raining when the sky is blue.
- Rain does not fall from a blue sky.
- A blue sky implies no rain.
- A blue sky is sufficient for it not to be raining.
- Not raining is necessary for a blue sky.

Many forms:

- If p, then q.
- p implies q.
- p only if q.
- **p is sufficient for q.**
- **q is necessary for p.**
- q if p.

The two in bold are very important in mathematics and science because they are very, very common and often read incorrectly. *Sufficient* follows the arrow in $p \to q$, and *necessary* works backward. We will see why when we examine the appropriate truth table.

Colloquially, we can refer to p as a premise or hypothesis and q as a conclusion. However, we will stop using those terms when we discuss logical deduction. Identifying p as a premise is useful for reasoning about $p \to q$, but it introduces ambiguity when we consider $\to$ as an operator. Just like with $\models$ and $\equiv$, additional symbols provide the appropriate context.

16.6.2 Defining $p \to q$

The truth table defining the conditional $\to$ is slightly surprising:

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

The first and last lines are expected. When truth implies truth or falsity implies falsity, the statement as a whole is true.

The second line also is expected. A true premise implying a false conclusion renders the statement false.

Now comes the surprise. A false hypothesis implying a *true* conclusion renders the statement as a whole *true*! This is only surprising because we are looking at one line at a time. Consider the general case where the premise is false. If you start reasoning from a false hypothesis, what is the result? Anything. So any time p is false, the statement as a whole is true.

We can break the conditional operator into core operators by listing the single negated output and applying De Morgan's laws:

$$\begin{aligned} \models p \to q &\equiv \neg(p \wedge \neg q) \\ &\equiv \neg p \vee q. \end{aligned}$$

So $p \to q$ is a true statement whenever the conclusion q is true or when we start from a false premise and $\neq p$ is true.

16.6.3 Converse, inverse, and contrapositive

Considering a few rules from arithmetic, we see that $\to$ is *not* commutative! The form $q \to p$ is the **converse** of $p \to q$, but the two statements are not equivalent. Similarly, we cannot simply negate terms to obtain the negation, so $\neg(p \to q)$ is not the same as the **inverse** $\neg p \to \neg q$. There is one other form here that *is* equivalent. The **contrapositive** $\neg q \to \neg q$ is equivalent to $p \to q$.

			converse:	**inverse:**	**contrapositive**
p	q	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
1	1	1	1	1	1
1	0	0	1	1	1
0	1	1	0	0	0
0	0	1	1	1	1

16.6.4 If and only if, or $\leftrightarrow$

One more form of the conditional is important because it is so frequently used, the phrase *if and only if.*

p if and only if q is a double conditional or **biconditional**. It means $p \to q \wedge q \to p$ and is written symbolically as $p \leftrightarrow q$. In text, *if and only if* often is abbreviated as **iff**.

Looking at its truth table, we see that $\leftrightarrow$ is the same as equivalence:

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
1	1	1	1	1
1	0	0	1	0
0	1	1	0	0
0	0	1	1	1

Which symbol you use is a matter of preference and emphasis.

16.7 Quantifiers

We are interested in two quantifiers for logical statements: *for all* and *there exists*. These are **universal** and **existential** quantifiers, respectively. To demonstrate these, I'm going to switch to **predicate logic**. Here properties have parameters, and the quantifiers describe possible values for those parameters. There is a third common quantifier, the **uniqueness** quantifier, but we will not explore it.

Say Dexter has fleas (which he doesn't). So far, we would simply associate this statement with a single variable. But to examine quantifiers, we need a bit more.

Let $P(p)$ be the property of having fleas. This $P(\text{Dexter})$ is a (false) statement that Dexter has fleas. We can generalize this to state that $\exists p \in D : P(p)$ where D is the set of dogs. If p is Dexter, then $P(p)$ would be stating that Dexter has fleas, so there does exist such a p. The colon (:) is read as "such that" or "satisfies" and sometimes is replaced by a vertical bar | or the backwards epsilon-ish symbol $\ni$.

But Dexter does not have fleas, so in reality $\exists p \in D : \neg P(p)$. Because Dexter is a dog, we know not *all* dogs have fleas. So $\exists p \in D : \neg P(p)$ is the same as $\neg \forall p \in D : P(p)$.

Translating to and from English needs attention to detail. The word *always* does not always translate into $\forall$. Some translations:

$P(p)$ is always true.	$\forall p : P(p)$.
$P(p)$ is almost always true.	$\exists p : \neg P(p)$.
There always is some way for $P(p)$ to be true.	$\exists p : P(p)$
Sometimes $P(p)$.	$\exists p : P(p)$.

And here are the reasons why we care about formalizing quantifiers in this class: **Negation is tricky**, and **composing or nesting quantifiers is tricky**.

16.7.1 Negating quantifiers

Symbolically, two rules apply:

- $\neg(\forall p : P(p))$ is the same as $\exists p : \neg P(p)$, and
- $\neg(\exists p : P(p))$ is the same as $\forall p : \neg P(p)$.

Reading the first aloud,

> Stating that not all p satisfy $P(p)$ is the same as saying there exists some p that satisfies $\neg P(p)$.

And the second,

> Stating that there does not exist a p such that $P(p)$ is the same as saying that all p satisfy $\neg P(p)$.

16.7.2 Nesting quantifiers

One other item to note: Different quantifiers are not operators, hence you cannot assume they commute! Quantifiers **nest**. As an example, consider two statements:

> All homework questions have been answered by at least one student.
> Some student has answered all homework problems.

Translating the first (true) statement into a symbolic form gives

$$\forall q \in \text{questions}\, \exists s \in \text{students} : s \text{ answered } q.$$

The second (false) statement becomes

$$\exists s \in \text{students}\, \forall q \in \text{questions} : s \text{ answered } q.$$

The symbolic versions appear similar. The only difference is the order of the quantifiers. But the statements obviously have very different meanings.

The text gives a nice visual interpretation in Section 3.5 using diagrams akin to Venn diagrams from set theory. Here we consider a more syntactic approach.

Given the English and logic statements:

> All homework questions have been answered by at least one student.
> $\forall q \in \text{questions}\, \exists s \in \text{students} : s \text{ answered } q$.

You can read the latter as the former, or as

> For all questions, there is some student who has answered that question.

Which student is meant depends on which question is considered.

Now consider the other statement:

> Some student has answered all homework problems.
> $\exists s \in \text{students}\, \forall q \in \text{questions} : s \text{ answered } q$.

Here the questions answered depends on which student is considered. And this statement says that *one* student has answered *all* questions.

For a more formal example, you read $\forall p \in \mathbb{J}^+ \exists q \in \mathbb{J}^+ : p < q$ as

> For all positive integers p there exists a positive integer q such that $p > q$.

The particular q depends on the p. There is not be a single q that works for all positive integers p. This statement is true over the integers.

Swapping the quantified statements produces a false statement. The translation of $\exists q \in \mathbb{J}^+ \forall p \in \mathbb{J}^+ : p < q$ is

> There exists a positive integer q such that for all positive integers p, $p < q$.

There is no such integer, as $q = q$ and thus $q \not< q$.

So we cannot simply swap quantifiers.

16.7.3 Combining nesting with negation

How do we negate the following true and equivalent English and logic statements?

> All homework questions have been answered by at least one person.
> $\forall q \in \text{questions}\, \exists s \in \text{students} : s \text{ answered } q$.

Consider working symbolically with the rules we already established:

$$\begin{aligned}
&\neg(\forall q \in \text{questions}\, \exists s \in \text{students} : s \text{ answered } q) \\
&\qquad \equiv \exists q \in \text{questions}\, \neg(\exists s \in \text{students} : s \text{ answered } q) \\
&\qquad \equiv \exists q \in \text{questions}\, \forall s \in \text{students} : \neg(s \text{ answered } q) \\
&\qquad \equiv \exists q \in \text{questions}\, \forall s \in \text{students} : s \text{ has not answered } q.
\end{aligned}$$

So the negation is

> There exists a question such that for all students the student has not answered the question.

Equivalently,

> There is some question where no student has answered that question.

Is this the same as just sticking a *not* in front of the original sentence?

> Not all homework questions have been answered by at least one person.

In this case, yes. But the other statement is not as simple.

Now consider the other statement:

> Some student has answered all homework problems.
> $\exists s \in \text{students}\, \forall q \in \text{questions} : s \text{ answered } q$.

Does the statement

> Not some student has answered all homework problems.

mean anything? Not really. So how can we negate this statement correctly?

Work with the symbolic form. Then

$$\begin{aligned}
\models \neg(\exists s \in &\text{students}\, \forall q \in \text{questions} : s \text{ answered } q) \\
&\equiv \forall s \in \text{students}\, \neg(\forall q \in \text{questions} : s \text{ answered } q) \\
&\equiv \forall s \in \text{students}\, \exists q \in \text{questions} : \neg(s \text{ answered } q) \\
&\equiv \forall s \in \text{students}\, \exists q \in \text{questions} : s \text{ has not answered } q.
\end{aligned}$$

So a correct negation is

> For all students, there is some question that student did not answer.

16.8 Logical deduction: Delayed until after the test

Now that we have *if-then* statements, we can speak more about *logical deduction.* And the simplest level, we can chain $\rightarrow$ operators to show deduction. But then we end up with statements like $((p \rightarrow q) \wedge (q \rightarrow r) \wedge \neg r) \rightarrow ((p \rightarrow r) \wedge \neg r) \rightarrow p$. These become unreadable quickly.

Instead, we introduce new symbols. The above could be written as $p \rightarrow q, q \rightarrow r, \neg r \vdash p \rightarrow r, \neg r \vdash p$. Taking up more room but being more clear, we can write the **logical argument** or **logical deduction** as

$$\begin{array}{l}
p \rightarrow q \\
q \rightarrow r \\
\neg r \\
\hline
\vdash p \rightarrow r \\
\neg r \\
\hline
\vdash \neg p
\end{array}$$

The general form in one line: Premise 1, premise 2 $\vdash$ conclusion. In table form,

$$\begin{array}{l}
\text{Premise 1} \\
\text{Premise 2} \\
\hline
\vdash \text{Conclusion.}
\end{array}$$

Both are read as *assuming premise one and premise two, infer the conclusion* or *premise one and premise two entail the conclusion.*

The $\vdash$ symbol is *syntactic sugar* meant to place emphasis where important. $p \vdash q$ is the same as $\vdash p \rightarrow q$ or just $p \rightarrow q$, but the former implies a deduction while the latter appears to be just a statement.

Sometimes three dots, $\therefore$, is used instead of $\vdash$. And sometimes (as in the text) neither appears in the tabular form. However, symbolic logic is about removing ambiguity that comes from language. Using the symbol helps disambiguate valid logical arguments from examples of invalid arguments or fallacies.

When reasoning about reasoning itself, the set of premises in a rule often are represented by Γ, so $\Gamma \vdash q$ is a rule with a *set* of premises Γ leading to a single result q.

Some forms or structures of logical arguments have classical names. These names came long before the symbolic form.

Modus ponens $p \to q, p \vdash q$, or

$$\begin{array}{l} p \to q \\ p \\ \hline \vdash q \end{array}$$

Modus tollens $p \to q, \neg q \vdash \neg p$, or

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \vdash \neg p \end{array}$$

Disjunctive syllogism $p \vee q, \neg p \vdash q$, or

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \vdash q \end{array}$$

Hypothetical syllogism (also called **transitivity**) $p \to q, q \to r \vdash p \to r$, or

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \vdash p \to r \end{array}$$

We also can write De Morgan's laws as laws of deduction, for example $\neg(p \vee q) \vdash \neg p \wedge \neg q$, or

$$\begin{array}{l} \neg(p \vee q) \\ \hline \vdash \neg p \wedge \neg q \ . \end{array}$$

Much of the reason why we care about the *correct* rules of deduction is to highlight incorrect rules, or **logical fallacies**. A list of a few common fallacies follows. Note that we include a symbol after the line in the tabular form; the negation of implication, $\nvdash$, lets you know we are talking about fallacies. The text does not use any symbol, so you may end up seeing these fallacies as valid.

Fallacy of the converse Given $p \to q$ and q, we cannot deduce p. Symbolically, $p \to q, q \nvdash p$ or

$$\begin{array}{l} p \to q \\ q \\ \hline \nvdash p \ . \end{array}$$

Fallacy of the inverse Given $p \to q$ and $\neg p$, we cannot deduce $\neg q$. Symbolically, $p \to q, \neg p \nvdash \neg q$ or

$$\begin{array}{l} p \to q \\ \neg p \\ \hline \nvdash \neg q \ . \end{array}$$

Fallacy of the alternative disjunct Given $p \vee q$ and p, we cannot deduce $\neg q$. Symbolically, $p \vee q, p \nvdash \neg q$ or

$$\begin{array}{l} p \vee q \\ p \\ \hline \nvdash \neg q \ . \end{array}$$

With the *exclusive-or* operator $\oplus$, we can conclude that only one disjunct is chosen. But the *or* operator $\vee$ allows both to occur at once.

Chapter 17

Homework for the fourth week: symbolic logic

17.1 Homework

Practice is absolutely critical in this class.

Groups are fine, turn in your own work. Homework is due in or before class on Mondays.

- Section 3.1
  - Problems 1-5
  - Problems 40, 42, 44
  - Problems 49-54
- Section 3.2
  - Problems 15-18
  - Problems 37-40
  - Problems 53-55
  - Problems 61, 62
- Section 3.3
  - Problems 1-5
  - Problems 13, 15, 20
  - Problems 35-38

- Problems 58, 60
- Problems 67, 68
- Problems 74, 75

- Section 3.4
  - Problems 1, 3, 6
  - Problem 51, 57, 58
- Section 3.1 again
  - Problems 55, 56
  - Problems 60-64
  - Problem 75
  - Problem 76. Hint: Quantifiers do not necessarily exclude each other.
- Negate the following, and decide if the statements are true or false.
  - There is a number p for all numbers q such that the difference between p and q is 2.
  - For all sets A, for all sets B, there is a set C such that $A \cap B = C$ and C is not $\emptyset$.
- Derive a logic expression from the following truth table. Attempt to simplify it remembering the distributive property, De Morgan's laws, and that $\models z \vee \neg z \equiv 1$ and $\models z \wedge \neg z \equiv 0$.

p	q	r	$f(p,q,r)$
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

- Section 3.6: **Delayed until after the test week.**
  - Problems 3, 6
  - Problems 17, 19, 21
  - Problems 47, 49

Note that you *may* email homework. However, I don't use Microsoft$^{\text{TM}}$ products (*e.g.* Word), and software packages are notoriously finicky about translating mathematics.

If you're typing it (which I advise just for practice in whatever tools you use), you likely want to turn in a printout. If you do want to email your submission, please produce a PDF or PostScript document.

Chapter 18

Solutions for fourth week's assignments

Also available as PDF.

18.1 Section 3.1

18.1.1 Problems 1-5

1. Logical statement: there is enough data to verify the statement.

2. Logical statement: again, there is enough data.

3. Not a logical statement: rhetorical, not logical.

4. Not a logical statement: this is a directive and not a statement.

5. Logical statement: verifiable.

18.1.2 Problems 40, 42, 44

40. He is not 48 years old.

42. She has green eyes, he is 48 years old, or both.

44. She has green eyes and he is not 48 years old.

18.1.3 Problems 49-54

49. $p \wedge \neg q$

50. $\neg p \vee \neg q$

51. $\neg p \vee q$

52. $q \wedge \neg p$

53. $\neg(p \vee q)$

54. $(p \vee q) \wedge \neg(p \wedge q)$

18.2 Section 3.2

18.2.1 Problems 15-18

15. $\models \neg(0 \wedge \neg 1) \equiv \neg(0 \wedge 0) \equiv \neg 0 \equiv 1.$

16. $\models \neg(\neg 0 \vee \neg 1) \equiv \neg(1 \vee 0) \equiv \neg 1 \equiv 0.$

17. $\models \neg\left[\neg 0 \wedge (\neg 1 \vee 0)\right] \equiv \neg(1 \wedge 0) \equiv \neg 0 \equiv 1.$

18. $\models \neg\left[(\neg 0 \wedge \neg 1) \vee \neg 1\right] \equiv \neg(0 \vee 0) \equiv 1.$

18.2.2 Problems 37-40

37. Two variables, so $2^2 = 4$ rows.

38. Three variables, $2^3 = 8$ rows.

39. Four variables, $2^4 = 16$ rows.

40. Five variables, $2^5 = 32$ rows.

18.2.3 Problems 53-55

53.

p	q	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \wedge (p \wedge q)$
1	1	1	1	1
1	0	1	0	0
0	1	0	0	0
0	0	1	0	0

54.

p	q	$\neg p \wedge \neg q$	$\neg p \vee q$	$(\neg p \wedge \neg q) \vee (\neg p \vee q)$
1	1	0	1	1
1	0	0	0	0
0	1	0	1	1
0	0	1	1	1

55.

p	q	r	$\neg p \wedge q$	$(\neg p \wedge q) \wedge r$
1	1	1	0	0
1	1	0	0	0
1	0	1	0	0
1	0	0	0	0
0	1	1	1	1
0	1	0	1	0
0	0	1	0	0
0	0	0	0	0

18.2.4 Problems 61, 62

61. Symbolically, the statement is $p \vee q$. So the negation is $\neg(p \vee q) \equiv \neg p \wedge \neg q$. Back into English: You cannot pay me now and you cannot pay me later.

62. Again, the statement is $\neg p \vee q$ and its negation is $p \wedge \neg q$. In English: I am going and she is not going.

18.3 Section 3.3

18.3.1 Problems 1-5

1. If it is breathing, then it must be alive.

2. If you see it on the Internet, then you can believe it.

3. If it is summer, then Lorri Morgan visits Hawaii.

4. If the number is Tom Shaffer's area code, then the number is 216.

(*Alternate from homeworks*: If the area code is 216, then it is Tom Shaffer's. *This is another reasonable interpretation. As is:* If you are Tom Shaffer, then your area code is 216.)

5. If it is a picture, then it tells a story.

18.3.2 Problems 13, 15, 20

13. If q is false, then the statement leads from a false premise and must be true. So assume q really is true. Then the validity of the statement depends on $(p \wedge q) \rightarrow q$ being true.

The clause $p \wedge 1$ simplifies, leading to $p \rightarrow 1$. From the truth table, we see that the value of p does not matter and the result always is true. So the statement is **true**.

15. The truth table for an if-then rule or conditional $\rightarrow$ has one false entry. The negation of the conditional thus has only one true entry and so is not a conditional itself. So the statement is **false**.

20. From a false hypothesis, anything can follow. The statement is **true**.

18.3.3 Problems 35-38

35. $\neg b \rightarrow \neg r$

36. $p \rightarrow \neg r$

37. $b \vee (p \rightarrow r)$

38. $p \wedge (r \rightarrow \neg b)$

18.3.4 Problems 58, 60

58.

p	q	$\neg q \rightarrow \neg p$	$(\neg q \rightarrow \neg p) \rightarrow \neg q$
1	1	1	0
1	0	0	1
0	1	1	0
0	0	1	1

60.

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
1	1	1	1	1
1	0	0	1	0
0	1	0	1	0
0	0	0	0	1

18.3.5 Problems 67, 68

67. That is an authentic Persian rug and I am not surprised.

68. Ella reaches that note and she does not shatter glass.

Quick note on Problem 67. Authentic Persian rugs and other rugs from that region ***must*** *have flaws in the design. That is a religious requirement. And you can tell if it's hand-made by checking the seams near the edges. If the seams are too small, the rug was machine-made.*

18.3.6 Problems 74, 75

74. The check is not in the mail or I am surprised.

75. She does or he will.

18.4 Section 3.4

18.4.1 Problems 1, 3, 6

1. Converse: If you were an hour, then beauty would be a minute. Inverse: If beauty were not a minute, then you would not be an hour. Contrapositive: If you were not an hour, then beauty is not a minute.

3. Converse: If you don't fix it, it ain't broke. Inverse: If it is broke, then you fix it. Contrapositive: If you fix it, it is broke.

6. Converse: If it contains calcium, then it is milk. Inverse: If it is not milk, then it does not contain calcium. Contrapositive: If it does not contain calcium, then it is not milk.

18.4.2 Problem 51, 57, 58

51. By the rules, this must be **contrary**.

57. $x = 1$ and $z = 37$. $x = 2$ and the Coen brothers are funny. There are no other rules governing these, so they must be consistent.

58. $x = 1$ and $x = 2$. $y = 1$ and $y = 2$. A variable cannot have two values, so this is contrary.

18.5 Section 3.1 again

18.5.1 Problems 55, 56

55. Literally translated, $\forall k \in \text{Items}\, \forall s \in \text{Stores} : \neg(k \text{ is available in } s)$. We can pull out the negation to see that this is the same as $\neg(\exists k \in \text{Items}\, \exists s \in \text{Stores} : k \text{ is available in } s$. So no item is available in any store. This is *not* the correct statement. The advertisement means to state that some items *may* not be available in all stores. We do not have the proper symbols to translate *may*.

56. The direct translation is $\forall p \in \text{People} : \neg(p \text{ has time to maintain his/her car properly})$. We can pull the negation out to see that $\neg(\exists p : p \text{ has time to maintain his/her car properly})$, and the original statement says that no one maintains his/her car properly. The intent likely was that *some* people do not have time to maintain their cars properly, or $\exists p : \neg(p \text{ has time to maintain his/her car properly})$.

18.5.2 Problems 60-64

60. A, B

61. A, C

62. C

63. B

64. A, C: "Not every" is both $\neg\forall x : P(x)$ and $\exists x : \neg P(x)$. Using the latter, $\exists p : \neg(\neg(F(p)))$, where $F(p)$ asserts that p has a frame. So the statement is $\exists p : F(p)$, or there is a framed picture.

18.5.3 Problem 75

The first statement is about *every* student, $\forall s : \neg(s \text{ passed the test})$

The second negates the entire thing, so there exists *some* student who did not pass, because $\models \neg\forall s : s \text{ passed the test} \equiv \exists s : s \text{ did not pass the test}$.

Note that the latter statement is still true if ***no*** *student passed. So you cannot infer that anyone passed in either statement.*

18.5.4 Problem 76

Hint: Quantifiers do not necessarily exclude each other.

Both statements are true but they have different quantifiers.

The original statement (for *some* real number) is true if you pick one number (say zero) and demonstrate its truth ($0^2 = 0 \geq 0$).

Your friend's statement applies to *all* real numbers. To demonstrate its validity, the statement must be proven to be true regardless of the value of x.

18.6 Negating statements

> **There is a number p for all numbers q such that the difference between p and q is 2.**

Symbolically,

$$\forall q \exists p : |p - q| = 2.$$

Note the order of the quantifiers. So the negation is

$$\neg(\forall q \exists p : |p - q| = 2) \equiv \exists q \forall p : |p - q| \neq 2.$$

And in English:

> For some number q, for all numbers p, the difference between p and q is not 2.

Quite often the phrasing is less bizarre if you push the negation only part-way through. Here, negating just the initial $\forall q$ gives the phrase:

> For some number q there is no number p such that the difference between p and q is 2.

The initial statement is true, and its negation is false.

> **For all sets A, for all sets B, there is a set C such that $A \cap B = C$ and C is not $\emptyset$.**

Translating:

$$\forall A \forall B \exists C : A \cap B = C \wedge C \neq \emptyset.$$

Pushing the negation through the quantifiers and applying De Morgan's laws,

$$\neg(\forall A \forall B \exists C : A \cap B = C \wedge C \neq \emptyset) \equiv \exists A \exists B \forall C : A \cap B \neq C \vee C = \emptyset.$$

Back into English:

> There is a set A for which there is a set B such that for all C, $A \cap B \neq C$ or $C = \emptyset$.

Note that the statement

$$\forall A \forall B \exists C : A \cap B = C \wedge C \neq \emptyset$$

is false. If two sets A and B are disjoint, then $A \cap B = \emptyset$. Thus the negation is true.

18.7 Function from truth table

Derive a logic expression from the following truth table. Attempt to simplify it remembering the distributive property, De Morgan's laws, and that $\models z \vee \neg z \equiv 1$ and $\models z \wedge \neg z \equiv 0$.

p	q	r	$f(p,q,r)$
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

There are fewer true values, so we start by listing individual conditions where the function is true. These are $p \wedge q \wedge r$, $p \wedge \neg q \wedge r$, and $p \wedge \neg q \wedge \neg r$. Joining these with *or*,

$$\models f(p,q,r) \equiv (p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r).$$

Using the distributive property, we can pull out a p. Then

$$\models f(p,q,r) \equiv p \wedge ((q \wedge r) \vee (\neg q \wedge r) \vee (\neg q \wedge \neg r)).$$

Now we can pull out either $\neg q$ or r to try simplifying inside the parenthesis. With the former:

$$\models f(p,q,r) \equiv p \wedge ((q \wedge r) \vee (\neg q \wedge (r \vee \neg r))).$$

Because $\models r \vee \neg r \equiv 1$,

$$\models f(p,q,r) \equiv p \wedge ((q \wedge r) \vee \neg q).$$

Now re-distribute $\neg q$ and simplify to see

$$\begin{aligned} \models f(p,q,r) &\equiv p \wedge (q \vee \neg q) \wedge (r \vee \neg q) \\ &\equiv p \wedge (r \vee \neg q). \end{aligned}$$

So a simplified form is $\models f(p,q,r) \equiv p \wedge (r \vee \neg q)$. Another equivalent form is $\models f(p,q,r) \equiv p \wedge (r \rightarrow q)$.

Chapter 19

Notes for the fifth week: review

Notes also available as PDF.

19.1 Review

Structure of the upcoming test:

- Ten questions. Chose ***six*** and solve them.
- Thus expect about seven minutes per question.
- Remember to read and answer the ***entire*** question.
- Closed book, *etc.* Calculators are fine but not necessary.
- Bring scratch paper and paper for writing up your results. Separately.
- Answers and explanations need to be indicated clearly.
- No questions are intended to be "trick" questions.
- Will cover the following topics:
  - inductive v. deductive reasoning with sequences,
  - problem solving,
  - set theory,
  - symbolic logic.
- Remember that solutions for the homework problems are available on-line: `http://jriedy.users.sonic.net/VI/math131-f08/`.

19.2 Inductive and deductive reasoning

Two primary forms of reasoning:

inductive Working from examples and intuiting how to extend them. Inductive reasoning does not prove anything.

deductive Extending hypotheses with rules to reach a conclusion. Deductive reasoning generates proofs (even if simple).

An example of *inductive reasoning*:

> It has been sunny all week, so it will be sunny tomorrow.

There are no explicit rules or assumptions. We just assume the pattern continues.

An example of *deductive reasoning*:

> The weather forecasts state that if the storm turns northward, it will rain tomorrow. The storm has turned northward, so it will rain tomorrow.

This sets up a rule from the weather forecast. Then the rule is applied to data, the storm turning northward, to reach a conclusion about tomorrow.

There rarely are clean-cut distinctions. Consider extending the sequence

$$35, 45, 55, \ldots$$

Reasoning *inductively*, we might assume that the numbers jump by ten. The next number will be 65, then 75, and so on.

Reasoning *deductively*, we assume this is an arithmetic sequence starting at 35 with an increment of 10. Under this assumption, the next two numbers will be $35 + 10 \cdot 3 = 65$ and $35 + 10 \cdot 4 = 75$.

The difference between the two forms is subtle. Deductive reasoning sets up explicit rules. The rules themselves may be discovered inductively, but making the rules specific and applying them carefully renders the result deductive.

19.3 Problem solving

Pólya's principles:

1. Understand the problem.
2. Devise a plan.
3. Carry out the plan.
4. Look back at your solution.

This is not a simple 1-2-3-4 recipe. Understanding the problem may include playing with little plans, or trying to carry out a plan may lead you back to trying to understand the problem.

19.3.1 Understand the problem

- Read the **entire** problem.

  Read the **whole** problem.

  Read **all of the** problem.

  One comment about the homeworks: Most people answer only part of any given problem.

- Determine what you **have** and what you **want**.

  To indicate an answer clearly to someone else (like me), you need to know what the answer is.

- Consider rephrasing the problem, either in English or symbolically.

  Rephrasing the problem may help you remember solution methods.

- Try some examples.

  This is close to devising a plan. Sometimes you may stumble upon an answer.

- Look for relationships between the data.

  Examples may help find relationships. The relationships may help you decide on a plan. Mathematics is about relationships between different entities; symbolic mathematics helps abstract away the entities themselves.

19.3.2 Devise a plan

Sometimes plans are "trivial," or so simple it seems pointless to make them specific. But write it out anyways. Often the act of putting a plan into words helps find flaws in the plan.

Try to devise a plan that you can check along the way. The earlier you detect a problem, the easier you can deal with it.

Some plans we've considered:

- Guessing and checking.

  Try a few combinations of the data. See what falls out. This is good for finding relationships and understanding the problem.

- Searching using a list.

  If you know the answer lies in some range, you can search that range systematically by building a list.

- Finding patterns.

  When trying examples, keep an eye open for patterns. Sometimes the patterns lead directly to a solution, and sometimes they help to break a problem into smaller pieces.

- Following dependencies / working backwards.

  Be sure to understand what results depend on which data. Look for dependencies in the problem. Sometimes pushing the data you have through all the dependencies will break the problem into simpler subproblems.

19.3.3 Carry out the plan

Attention to detail is critial here.

When building a list, be sure to carry out a well-defined procedure. Or when looking for patterns, be systematic in the examples you try. Don't jump around randomly.

19.3.4 Look back at your solution

Can you check your result? Sometimes trying to check reveals new relationships that could lead to a better solution.

Think about how your solution could help with other problems.

19.4 Sequences

A sequence is an ordered list of numbers. Two common kinds are:

arithmetic Adds a constant increment at each step.

geometric Multiplies by a constant at each step.

One method for extending a sequence is through **successive differences**. Consider the sequence

$$11 \quad 22 \quad 39 \quad 64 \quad \ldots$$

To compute the next term, form differences until you find a constant column:

i	A_i	$\Delta_i^{(1)} = A_i - A_{i-1}$	$\Delta_i^{(2)} = \Delta_i^{(1)} - \Delta_{i-1}^{(1)}$
1	11		
2	22	11	
3	39	17	6
4	62	23	6
5	**91**	**29**	**6**

19.5 Set theory

set An **unordered** collection of **unique** elements.

You can write a set by listing its entries, $\{1, 2, 3, 4\}$, or through set builder notation, $\{x \mid x \text{ is a positive integer}, x < 5\}$.

empty set The **unique** set with no elements: $\{\}$ or $\emptyset$. Be sure to know the relations between the empty set and other sets, and also how the empty set behaves in operations.

element of You write $x \in A$ to state that x is an element of A. The symbol is not an "E" but is almost a Greek ϵ. Think of a pitchfork.

Note that $1 \in \{1, 2\}$ and $\{1\} \in \{\{1\}, \{2\}\}$, but $\{1\} \notin \{1, 2\}$.

subset Given two sets A and B, $A \subset B$ if every element of A is also an element of B. So $A \subset B$ is equivalent to $x \in A \to x \in B$.

One implication is that $\emptyset \subset A$ for all sets A. This statement is **vacuously** true.

Here $\{1\} \subset \{1, 2\}$ and $\{1\} \not\subset \{\{1\}, \{2\}\}$.

superset Given two sets A and B, $B \supset A$ if every element of A is also an element of B.

proper subset or superset A subset or superset relation is **proper** if it implies the sets are not equal. An equivalent symbolic logic statement would be $(x \in A \to x \in B) \wedge (\exists x \in B : x \notin A)$.

Venn diagram A blobby diagram useful for illustrating operations and relations between two or three sets.

union $A \cup B = \{x \mid x \in A \vee x \in B\}$. The union contains all elements of both sets.

intersection $A \cap B = \{x \mid x \in A \wedge x \in B\}$. The intersection contains only those elements that exist in both sets.

set difference $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$. The set difference contains elements of the first set that are **not** in the second set. It cannot contain any elements of the second set.

You can use symbolic logic to write the result of multiple operations.

$$\begin{aligned}(A \cap B) \cup C &= \{x \mid x \in A \wedge x \in B\} \cup C \\ &= \{x \mid (x \in A \wedge x \in B) \vee x \in C\}.\end{aligned}$$

19.6 Symbolic logic

logical statement Some clear statement that is either true or false.

Some different ways of writing true or false are acceptable:

true	false
T	F
1	**0**
$\top$	$\bot$

The test's questions use 1 and 0.

logical variable A variable standing for some logical statement. Common variables are p, q, r.

truth table A systematic listing of all possible input truth values for an expression.

negation True when the variable is false and false when the variable is true. Will be written $\neg p$.

and True only when all variables are true. Will be written $p \wedge q$.

or False only when all variables are false. Will be written $p \vee q$.

equivalence The logical form of equality. Will be written $p \equiv q$.

conditional If p then q. True whenever true implies true or when false implies anything. Will be written $p \to q$.

tautology A statement that always is true. Will be written $\models$, as in $\models p \vee \neg p$. This is just for emphasis; there is no real difference with $(p \vee \neg p) \equiv 1$.

A truth table defining four operations above:

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \to q$
1	1	0	1	1	1
1	0	0	0	1	0
0	1	1	0	1	1
0	0	1	0	0	1

Note that $\neg p$ did not need all four lines. It does not depend on q and has the same value regardless of whether q is false or true.

19.6.1 From truth tables to functions

Consider the truth table:

p	q	$f(p,q)$
1	1	1
1	0	0
0	1	1
0	0	0

We can derive an expression for $f(p,q)$ in two ways. Obviously, any expressions must simplify to q.

One method is to work from the true values. You *or* together *and* expressions. There is one *and* expression per true value. In this case, we have $\models f(p,q) \equiv (p \wedge q) \vee (\neg p \wedge q)$. Pulling out the q, this simplifies $\models f(p,q) \equiv (p \vee \neg q) \wedge q \equiv 1 \wedge q \equiv q$.

The other method is to work from the false values. You *and* together *or* expressions. There is one *or* expression per false value. Here, $\models f(p,q) \equiv (\neg p \vee q) \wedge (p \vee q)$. Pulling out q again, $\models f(p,q) \equiv (\neg p \wedge p) \vee q \equiv 0 \vee q \equiv q$.

19.6.2 De Morgan's laws and forms of conditionals

De Morgan's laws are two very useful methods for negating terms symbolically:

$$\begin{gathered} \models \neg(p \vee q) \equiv \neg p \wedge \neg q, \text{ and} \\ \models \neg(p \wedge q) \equiv \neg p \vee \neg q. \end{gathered}$$

As an example, consider negating the conditional. Use the equivalent form $\models p \rightarrow q \equiv \neg p \vee q$. Then

$$\begin{aligned} \neg(p \rightarrow q) &= \neg(\neg p \vee q) \\ &= \neg(\neg p) \wedge \neg q \\ &= p \wedge \neg q. \end{aligned}$$

So the negation of a conditional is **not** a conditional itself.

There are four related forms of conditional:

conditional $p \rightarrow q$: If you grew up in Alaska, you have seen snow.

inverse $\neg p \rightarrow \neg q$: If you did not grow up in Alaska, you have not seen snow.

converse $q \rightarrow p$: If you have seen snow, you grew up in Alaska.

contrapositive $\neg q \rightarrow \neg p$: If you have not seen snow, you did not grow up in Alaska.

Only the **contrapositive** has the same meaning as the original conditional, so $\models p \to q \equiv \neg q \to \neg p$.

The **converse** and **inverse** are related to each other but are **not** equivalent to the original conditional. The inverse is the contrapositive of the converse: $\models q \to p \equiv \neg p \to \neg q$

19.6.3 Quantifiers

quantifier A statement regarding some or all possible entries of some set.

existential Declares that some entry exists, so $\exists x : x \in A$ states that A is not empty.

universal Declares some property is true for every value. So $\forall x \in A : x \in B$ is another way of writing $A \subset B$.

predicate Or **property**. A symbolic way of expressing that some property holds. For example, understands(s, t) may state that student s understands topic t. A less obtuse but still acceptable statement for a simple predicate is just "s understands q."

The translation of phrases from English to quantified symbolic logic can be tricky.

Almost every student understands all symbolic logic topics.

can translate to

$$\exists s\, \forall t : \neg\, \text{understands}(s, t)$$

because we don't measure *how many* but rather that there is or is not one.

19.6.4 Nesting and negating quantifiers

Nested quantifiers are not operators. Each quantifier applies to the entire remaining statement. $\forall s \exists t$ states that for every s, there exists a t **for that s**. Meanwhile, $\exists t \forall s$ states that there exists one single t for every and all s.

Two rules for negating quantifiers:

$\neg \forall s : P(s)$ is the same as $\exists s : \neg P(s)$, and

$\neg \exists s : P(s)$ is the same as $\forall s : \neg P(s)$.

So saying "not all" is the same as "there exists one for not", and saying "there does not exist" is the same as "for all, not".

As an example, we negate the statement above,

Almost every student understands all symbolic logic topics.

and its translation

$$\exists s \, \forall t : \neg \, \text{understands}(s, t).$$

The symbolic negation is

$$\begin{aligned}
\neg(\exists s \, \forall t : \neg \, \text{understands}(s, t)) &= \forall s \, \neg(\forall t : \neg \, \text{understands}(s, t)) \\
&= \forall s \, \exists t : \neg(\neg \, \text{understands}(s, t)) \\
&= \forall s \, \exists t : \text{understands}(s, t).
\end{aligned}$$

Translating back to English,

> All students understand some symbolic logic topic.

It may be true that no two students understand the *same* topic, but every student understands some topic.

Chapter 20

First exam and solutions

Available as PDF.

Chapter 21

Notes for the sixth week: numbers and computing

Notes also available as PDF.

What we will cover from Chapter 4:

- Numbers and digits in different bases, with historical context
- Arithmetic, digit by digit

And additionally, I'll give a brief summary of computer arithmetic.

21.1 Positional Numbers

A number is a concept and not just a sequence of symbols. We will be discussing ways to express numbers.

Multiple *types* of numbers:

nominal A *nom*inal number is just an identifier (or *name*). In many ways these are just sequences of symbols.

ordinal An *ord*inal number denotes *order*: 1^{st}, 2^{nd}, ...

Adding ordinal or nominal numbers doesn't make sense. This brings up a third type:

cardinal Cardinal numbers *count*.

The name comes from the *cardinality* of sets.

Before our current form:

- Piles of rocks don't work well for merchants.
- **Marks** on sticks, then marks on papyrus.

Marking numbers is costly. A large number becomes a large number of marks. Many marks lead to many errors. Merchants don't like errors. So people started using symbols rather than plain marks.

An intermediate form, **grouping**:

- Egyptian: Different symbols for different levels of numbers: units, tens, hundreds. Grouping within the levels.
- Roman: Symbols for groups, with addition and subtraction of symbols for smaller groups.
- Greek (and Hebrew and Arabic): Similar, but using all their letters for many groups.
- Early Chinese: Denote the number of marks in the group with a number itself. . .

Getting better, but each system still has complex rules. The main problems are with skipping groups. We now use zero to denote an empty position, but these systems used varying amounts of space. Obviously, this could lead to trade disagreements. Once zeros were adopted, many of these systems persisted in trade for centuries.

Now into forms of positional notation, shorter and more direct:

- Babylonian:
  - Two marks, tens and units.
  - Now the marks are placed by the number of 60s.
  - Suffers from complicated rules about zeros.
  - (Using 60s persists for keeping time...)
- Mayan:
  - Again, two kinds of marks for fives and units.
  - Two positional types: by powers of 20, and by powers of 20 except for one power of 18.
  - (Note that $18 \cdot 20 = 360$, which is much closer to a year.)
  - Essentially equivalent to what we use, but subtraction in Mayan is much easier to see.
- (many other cultures adopted similar systems (*e.g.* Chinese rods)

Current: **Hindu-Arabic numeral system**

The characters differ between cultures, but the idea is the same. The characters often are similar as well. Originated in the region of India and was carried west through trade. No one knows when zero was added to the digits. The earliest firm evidence is in Arab court records regarding a visitor from India and a description of zero from around 776 AD. The first inscription found with a zero is from 876 AD in India. However, the Hindu-Arabic system was not adopted outside mathematics even in these cultures. Merchants kept to a system similar to the Greek and Hebrew systems using letters for numbers.

Leonardo Fibonacci brought the numerals to Europe in the 13$^{\text{th}}$ century (after 1200 AD) by translating an Arabic text to Latin. By 15$^{\text{th}}$ century, the numeral system was in wide use in Europe. During the 19$^{\text{th}}$ century, this system supplanted the rod systems in Asia.

The final value of the number is based on the positions of the digits:

$$1234 = 1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0.$$

We call ten the **base**. Then numbers becomes polynomials in terms of the base b,

$$1234 = b^3 + 2 \cdot b^2 + 3 \cdot b^1 + 4.$$

Here $b = 10$.

So we moved from marks, where 1000 would require 1000 marks, to groups, where 1000 may be a single mark but 999 may require dozens of marks. Then we moved to positional schemes where the number of symbols depends on the *logarithm* of the value; $1000 = 10^3$ requires $4 = 3 + 1$ symbols.

After looking at other bases, we will look into operations (multiplication, addition, *etc.*) using the base representations.

21.2 Converting Between Bases

Only three bases currently are in wide use: base 10 (decimal), base 2 (binary), and base 16 (hexadecimal). Occasionally base 8 (octal) is used, but that is increasingly rare. Other conversions are useful for practice and for seeing some structure in numbers. The structure will be useful for computing.

Before conversions, we need the digits to use. In base b, numbers are expressed using digits from 0 to $b - 1$. When b is past 10, we need to go beyond decimal numerals:

Value:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Digit:	0	1	2	3	4	5	6	7	8	9	**A**	**B**	**C**	**D**	**E**	**F**

Upper- and lower-case are common.

So in hexadecimal, $\mathsf{DECAFBAD}$ is a perfectly good number, as is $\mathsf{DEADBEEF}$. If there is a question of what base is being used, the base is denoted by a subscript. So 10_{10} is a decimal ten and 10_2 is in binary.

To find values we recognize more easily, we convert to decimal. Then we will convert *from* decimal.

21.2.1 Converting to Decimal

Converting to decimal using decimal arithmetic is straight-forward. Remember the expansion of 1234 with base $b = 10$,

$$\begin{aligned}1234 &= 1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0 \\ &= b^3 + 2 \cdot b^2 + 3 \cdot b^1 + 4.\end{aligned}$$

Each digit of DEAD has a value, and these values become the coefficients. Then we expand the polynomial with $b = 16$. In a straight-forwart way,

$$\begin{aligned}\mathsf{DEAD} &= \mathsf{D} \cdot 16^3 + \mathsf{E} \cdot 16^2 + \mathsf{A} \cdot 16^1 + \mathsf{D} \\ &= 13 \cdot 16^3 + 14 \cdot 16^2 + 10 \cdot 16 + 13 \\ &= 13 \cdot 4096 + 14 \cdot 256 + 10 \cdot 16 + 13 \\ &= 57005.\end{aligned}$$

We an use **Horner's rule** to expand the polynomial in a method that often is faster,

$$\begin{aligned}\mathsf{DEAD} &= ((13 \cdot 16 + 14) \cdot 16 + 10) \cdot 16 + 13 \\ &= (222 \cdot 16 + 10) \cdot 16 + 13 \\ &= 3562 \cdot 16 + 13 \\ &= 57005.\end{aligned}$$

Let's try a binary example. Convert 1101_2 to decimal:

$$\begin{aligned}1101_2 &= (((1 \cdot 2 + 1) \cdot 2 + 0) \cdot 2 + 1 \\ &= (3 \cdot 2 + 0) \cdot 2 + 1 \\ &= 6 \cdot 2 + 1 \\ &= 13.\end{aligned}$$

Remember the rows of a truth table for two variables? Here,

$$\begin{aligned}11_2 &= 2 + 1 = 3, \\ 10_2 &= 2 + 0 = 2, \\ 01_2 &= 0 + 1 = 1, \text{ and} \\ 00_2 &= 0 + 0 = 0.\end{aligned}$$

21.2.2 Converting from Decimal

To convert to binary from decimal, consider the previous example:

$$\begin{aligned}
13 &= 8 + \mathbf{5} \\
&= 8 + 4 + \mathbf{1} \\
&= 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 \\
&= 1101_2.
\end{aligned}$$

At each step, we find the largest power of two less than the remaining number. Another example for binary:

$$\begin{aligned}
293 &= 256 + \mathbf{37} \\
&= 256 + 32 + \mathbf{5} \\
&= 256 + 32 + 4 + 1 \\
&= 1 \cdot 2^8 + 1 \cdot 2^5 + 1 \cdot 2^2 + 1 \\
&= 100100101_2.
\end{aligned}$$

And in hexadecimal,

$$\begin{aligned}
293 &= 256 + \mathbf{37} \\
&= 1 \cdot 256 + 2 \cdot 16 + 5 \\
&= 125_{16}.
\end{aligned}$$

You can see why some people start remembering powers of two.

If you have no idea where to start converting, remember the relations $b^{\log_b x} = x$ and $\log_b x = \log x / \log b$. Rounding $\log_b x$ up to the larger whole number gives you the number of base b digits in x.

The text has another version using remainders. We will return to that in the next chapter. And conversions to and from binary will be useful when we discuss how computers manipulate numbers.

21.3 Operating on Numbers

Once we split a number into digits (decimal or binary), operations can be a bit easier.

We will cover multiplication, addition, and subtraction both

- to gain familiarity with positional notation, and
- to compute results more quickly and mentally.

Properties of positional notation will help when we explore number theory.

We will use two properties frequently:

- Both multiplication and addition **commute** ($a + b = b + a$) and re-**associate** $(a + b) + c = a + (b + c)$.
- Multiplication **distributes** over addition, so $a(b + c) = ab + ac$.
- Multiplying powers of a common base adds exponents, so $b^a \cdot b^c = b^{a+c}$.

21.3.1 Multiplication

Consider multiplication. I once had to learn multiplication tables for 10, 11, and 12, but these are completely pointless.

Any decimal number multiplied by 10 is simply shifted over by one digit,

$$\begin{aligned} 123 \cdot 10 &= (1 \cdot 10^2 + 2 \cdot 10^1 + 3 \cdot 10^0) \cdot 10 \\ &= 1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10^1 \\ &= 1230. \end{aligned}$$

Multiplying by $11 = 1 \cdot 10 + 1$ is best accomplished by adding the other number to itself shifted,

$$123 \cdot 11 = 123 \cdot (10 + 1) = 1230 + 123 = 1353.$$

And for $12 = 1 \cdot 10 + 2$, you double the number,

$$123 \cdot 12 = 123 \cdot (10 + 2) = 1230 + 246 = 1476.$$

Multiplying longer numbers quickly follows the same pattern of shifting and adding. We can expand $123 \cdot 123 = 123 \cdot (1 \cdot 10^2 + 2 \cdot 10 + 3)$ to

$$\begin{array}{r} 123 \\ \times\ 123 \\ \hline 369 \\ 2460 \\ 12300 \\ \hline 15129 \end{array}$$

Another method expands the product of numbers as a product of polynomials, working one term at a time. This is essentially the same but not in tabular form:

$$\begin{aligned} 123 \cdot 123 &= (1 \cdot 10^2 + 2 \cdot 10 + 3) \cdot (1 \cdot 10^2 + 2 \cdot 10 + 3) \\ &= (1 \cdot 10^2 + 2 \cdot 10 + 3) \cdot (1 \cdot 10^2 + 2 \cdot 10) + (1 \cdot 10^2 + 2 \cdot 10 + 3) \cdot 3 \\ &= (1 \cdot 10^2 + 2 \cdot 10 + 3) \cdot (1 \cdot 10^2 + 2 \cdot 10) + (3 \cdot 10^2 + 6 \cdot 10 + 9) \\ &= \ldots \end{aligned}$$

This form splits the sums apart as well; we will cover that next.

Bear in mind that short-term memory is limited to seven to eight pieces of information. Structure mental arithmetic to keep as few pieces in flight as possible. One method is to break multiplication into stages. In long form, you can group the additions. For example, expanding $123 \cdot 123 = 123 \cdot (1 \cdot 10^2) + (123 \cdot 23) = 123 \cdot (1 \cdot 10^2) + (123 \cdot 2 \cdot 10 + 123 \cdot 3)$,

$$\begin{array}{r} 123 \\ \times\ 123 \\ \hline 369 \\ 2460 \\ \hline 2829 \\ 12300 \\ \hline 15129 \end{array}$$

Assuming a small number uses only one slot in your short-term memory, need track only where you are in the multiplier, the current sum, the current product, and the next sum. That leaves three to four pieces of information to use while adding.

One handy trick for 15% tips: divide by ten, divide that amount by two, and add the pieces. We can use positional notation to demonstrate how that works,

$$\begin{aligned} x \cdot 15\% &= (x \cdot 15)/100 \\ &= ((x \cdot (10 + 5))/100 \\ &= ((x \cdot 10) + (x \cdot (10/2)))/100 \\ &= x/10 + (x/10)/2 \end{aligned}$$

21.3.2 Addition

Digit-by-digit addition uses the commutative and associative properties:

$$\begin{aligned} 123 + 456 &= (1 \cdot 10^2 + 2 \cdot 10 + 3) + (4 \cdot 10^2 + 5 \cdot 10 + 6) \\ &= (1 + 4) \cdot 10^2 + (2 + 5) \cdot 10 + (3 + 6) \\ &= 579. \end{aligned}$$

Naturally, when a digit threatens to roll over ten, it **carries** to the next digit. Expanding the positional notation,

$$\begin{aligned} 123 + 987 &= (1 \cdot 10^2 + 2 \cdot 10 + 3) + (9 \cdot 10^2 + 8 \cdot 10 + 7) \\ &= (1 + 9) \cdot 10^2 + (2 + 8) \cdot 10 + (3 + 7) \\ &= 10 \cdot 10^2 + 10 \cdot 10 + 10. \end{aligned}$$

Because the coefficients are greater than $b - 1 = 9$, we expand those coefficients. Commuting and reassociating,

$$\begin{aligned}
123 + 987 &= 10 \cdot 10^2 + 10 \cdot 10 + 10 \\
&= (1 \cdot 10 + 0) \cdot 10^2 + (1 \cdot 10 + 0) \cdot 10 + (1 \cdot 10 + 0) \\
&= 1 \cdot 10^3 + 1 \cdot 10^2 + 1 \cdot 10 + 0 \\
&= 1110.
\end{aligned}$$

However, when working quickly, or when the addition will be used in another operation, you do not need to expand the carries immediately. This is called a **redundant representation** because numbers now have multiple representations. You can represent 13 as $1 \cdot 10 + 3$ or simply as 13.

If you work that way mentally, you need to keep the intermediate results in memory. So during multiplying, you only need to work out the carries every three to four digits. . .

21.3.3 Subtraction

In systems with signed numbers, we know that subtracting a number is the same as adding its negation: $a - b = a + (-b)$. So we expect the digit-by-digit method to work with each digit subtracted, and it does. Because $-a = -1 \cdot a$, we can distribute the sign over the digits:

$$\begin{aligned}
456 - 123 &= (4 \cdot 10^2 + 5 \cdot 10 + 6) - (1 \cdot 10^2 + 2 \cdot 10 + 3) \\
&= (4 \cdot 10^2 + 5 \cdot 10 + 6) + (-(1 \cdot 10^2 + 2 \cdot 10 + 3)) \\
&= (4 \cdot 10^2 + 5 \cdot 10 + 6) + (-1 \cdot 10^2 + -2 \cdot 10 + -3) \\
&= (4 - 1) \cdot 10^2 + (5 - 2) \cdot 10 + (6 - 3) \\
&= 333.
\end{aligned}$$

As with carrying, **borrowing** occurs when a digit goes negative:

$$\begin{aligned}
30 - 11 &= (3 \cdot 10^1 + 0) - (1 \cdot 10^1 + 1) \\
&= (3 - 1) \cdot 10^1 + (0 - 1) \\
&= 2 \cdot 10^1 + -1 \\
&= 1 \cdot 10^1 + (10 - 1) \\
&= 1 \cdot 10^1 + 9 \\
&= 19.
\end{aligned}$$

Again, you can use a redundant intermediate representation of $2 \cdot 10^1 - 1$ if you're continuing to other operations. And if **all** the digits are negative, you can factor out -1,

$$\begin{aligned}
123 - 456 &= (1 \cdot 10^2 + 2 \cdot 10 + 3) - (4 \cdot 10^2 + 5 \cdot 10 + 6) \\
&= (1-4) \cdot 10^2 + (2-5) \cdot 10 + (3-6) \\
&= (-3) \cdot 10^2 + (-3) \cdot 10 + (-3) \\
&= -(3 \cdot 10^2 + 3 \cdot 10 + 3) \\
&= -333.
\end{aligned}$$

21.3.4 Division and Square Root: Later

We will cover these later with number theory.

21.4 Computing with Circuits

No one can argue that computing devices (computers, calculators, medical monitors, *etc.*) are irrelevant to everyday life. Here we lay the groundwork for how computers compute.

Essentially, computers perform arithmetic on binary numbers. But different methods of combining the arithmetic operations produce character strings, sounds, graphics, ...

While those are courses in themselves, we at least can explain the very lowest levels of computer arithmetic. Automated computing is in its relative infancy. People have been building roads, bridges, and vehicles for thousands of years. Even motors are hundreds of years old. But modern computing is less than a hundred years old and became wide-spread only 30 years ago. Before the 1970s, desktop *calculators* were rare. And before the 1980s, calculators were virtually unaffordable.

Maybe someday we will be able to take safe computing for granted just like we take safe bridges for granted, but not yet. It's important at least to have heard how computing works so you can gain a sense of where limitations are. Consider an issue like the largest range of numbers you can represent exactly in a calculator, spreadsheet, or other program. Each may have different limitations that appear random but certainly are not. Having some sense of how computers compute lets you explain or (hopefully) anticipate limitations and work around them.

21.4.1 Representing Signed Binary Integers

Converting non-negative numbers to binary is straight-forward. Computer representations work with a limitied number of **b**inary dig**it**s, or **bit**s. With

32 bits, any non-negative *integer* less than $2^{33} = 8589934592 \approx 10^{9.9}$ can be represented exactly. With n bits, all non-negative integers less than 2^{n+1} can be represented exactly. For example, the largest two bit number is $11_2 = 3 < 2^2 = 4$.

Representing both positive and *negative* numbers, however, presents some design choices. One can use one of the bits (often the leading bit) as a sign bit. The number then becomes $-1^{\text{sign bit}} \cdot$ the rest of the bits. This reduces the representable range of n bits to $(-(2^n), 2^n)$ and requires treating one bit specially during operations. (The notation (a, b) is an **open range**, one that does not include its endpoints.) We need separate operations for $a + b$ and $a - b$. Also, we need to cope with $+0$ and -0.

We can eliminate the need for separate operations and also eliminate the signed zero.

A representation named **one's complement** plays a little trick with arithmetic to absorb the sign into the number. This allows using addition for subtraction...

We start by **negating** a number if it is negative:

#	Bits
3	01 1
2	01 0
1	00 1
0	00 0
-0	11 1
-1	11 0
-2	10 1
-3	10 0

Adding two n-digit numbers may produce an $n + 1$-digit result. For example, $11_2 + 11_2 = 110_2$ in binary or $5 + 6 = 11$ in decimal. Consider three bit addition:

$$\begin{array}{r} 110 \\ + \quad 10 \\ \hline \mathbf{1}000 \end{array}$$

If we capture the carry bit **1** and feed it back around, then $110_2 + 10_2 \mapsto 000_2 + 1_2 = 1_2$. In one's complement, this is $-1 + 2 = 1$ as expected.

So to add two numbers, *positive or negative*, we just add the one's complement representation. To subtract $a - b$, we negate b and add it to a. **We only need one operation, addition, for addition and subtraction.**

But we still have given an entire bit over to the sign. We can do slightly better with **two's complement.** More importantly, we can reduce the system to having only a single, unsigned zero. Having an unsigned zero is much easier to handle with multiplication and division.

To represent a negative number in two's complement, we negate it and add one:

#	Bits
3	011
2	010
1	001
0	000
-1	111
-2	110
-3	101
-4	111

By not including -0, we have room for one more number. By the two's complement method, it happens to fall on the end of the negative scale. Here, n bits represent all integers in $[-2^n, 2^n)$. (The notation $[a, b]$ is a **closed range** including a and b. Notations using square brackets on one side but not the other are half-open and include the end-point against the square bracket.)

There are other representations:

- A **biased** representation adds 2^{n-1} or $2^{n-1} - 1$ to every number and then represents the result. This shifts all the negative numbers to be non-negative. This representation has an explicit sign bit but only a single zero.
- A **base -2** rather than base 2 representation is bizarre, but it works. These most often are used for redundant representations inside other arithmetic operations. There are twice as many negative numbers as positive numbers, no sign bit, and only a single zero.
- Larger bases can be used by grouping bits. This also allows for more redundant representations. One representation using 1, 0, and -1 for digits is particularly interesting, but we won't cover it here.

21.4.2 Adding in Binary with Logic

Above we have reduced addition and subtraction of signed numbers into simple addition. Here we implement addition in logic and construct the **half adder** and **full adder** circuits.

Consider a truth table for $a \wedge b$ and $a \oplus b$ (*exclusive or*):

a	b	$a \wedge b$	$a \oplus b$	$a + b$
1	1	1	0	10_2
1	0	0	1	01_2
0	1	0	1	01_2
0	0	0	0	00_2

If we append a column representing the sum of a and b in binary, we see that the first digit is $a \wedge b$ and the second is $a \oplus b$!

This is a **half adder**. The half adder takes two bits as input and produces a sum bit $s = a \oplus b$ and a carry bit $c = a \wedge b$.

(drawing)

A **full adder** takes input bits and a previous carry bit to produce an output sum and carry. We can add $a + b$ and then $(a + b) + c_{\text{in}}$. Note that only one of those sums can generate a carry, so *or*-ing the carry outputs generates the final output. $1 + 1 + 1 = 3 = 11_2 < 100_2$, so the sum's output cannot require more than two bits.

So a full adder can be constructed with two half-adders and one extra or-gate for the carry:

a	b	c_{in}	$(a \oplus b) \oplus c_{\text{in}}$	$(a \wedge b) \vee (c_{\text{in}} \wedge (a \oplus b))$
1	1	1	1	1
1	1	0	1	0
1	0	1	1	0
1	0	0	0	1
0	1	1	1	0
0	1	0	0	1
0	0	1	0	1
0	0	0	0	0

To add two n bit numbers, you start by adding the low-order bits (coefficient in front of 2^0) with a half-adder. The sum is output and the carry follows into a full adder for adding the coefficients of 2^1. The process continues resulting in an n-bit sum and a single carry bit.

The carry bit often is ignored, leading to **overflow** and **wrap-around**. At a low level, adding two positive integers each greater than 2^{n-1} produces a *negative number*! This is terribly handy for some algorithms and detrimental to others. All architectures make the carry bit available for diagnosing overflow, but not all programming environments let users access that information.

Adding two n-bit numbers requires a minimum of one half-adder and $n - 1$ full adders, or $2n - 1$ half-adders and $n - 1$ *or* gates, or $2n - 1$ *exclusive-or* gates, $2n - 1$ *and* gates, and $n - 1$ *or* gates. Because of the dependence on the previous bit sum's carry output, it appears that each bit must be computed one at a time, or **serially**. There are tricks using redundant representations that allow computing the result in larger chunks, exposing more **parallelism** within the logic gates.

21.4.3 Building from Adders

Given addition, we could implement multiplication as *repeated addition.* Remember the Egyptian algorithm from the text?

The binary representation of the multiplier serves as a **mask**. Consider multiplying the 3-bit numbers $011_2 = 3$ and $101_2 = 5$:

$$\begin{array}{r|l} 011 & \cdot 1 \\ 011 & \cdot 0 \\ 011 & \cdot 1 \\ \hline 01111 & \end{array}$$

At each step, the bits of 101_2 determine whether or not a shifted copy of 011_2 is added into the result. We can implement this by shifting and adding serially, or we can construct a **multiplier array** out of adders.

Again, there are optimizations related to redundant representations, but ultimately most processors dedicate a large amount of their physical size (and "power budget") to multiplier arrays.

The problem of overflow becomes very important for multiplication. Because $2^n \cdot 2^n = 2^{2n}$, the product of two n-bit numbers may require $2n$ bits. Most architectures deliver the result in two n-bit **registers** (the limited number of variables a processor has to work with).

21.4.4 Decimal Arithmetic from Binary Adders

Ok, so we can add, subtract, and multiply numbers in binary. What about decimal? Alas, we lack the nifty two's complement tricks in decimal, so all decimal units need to cope with signs differently. Most use explicit signs and always convert -0 to 0.

For integers, conversion back and forth can occur exactly as in class. With 32 bits, there are at most $\lceil 32 \cdot \log_{10} 2 \rceil = 10$ decimal digits. (The notation $\lceil x \rceil$ rounds x to the closest integer $k > x$.) So software can lop off digits one at a time, often using the text's algorithm with remainders.

There are times when you want to work directly with decimal numbers, however. Some of these are dictated by legal or engineering considerations. For example, the "cpu" of a hand-held calculator is does not really run software or store many intermediate results. There, every result is calculated in decimal often using a representation called **BCD** for **b**inary **c**oded **d**ecimal.

A decimal number is represented digit-by-digit in binary. So $29 = 2 \cdot 10 + 9 = (10_2) \cdot 10 + (1001_2)$. This is relatively inefficient. The largest two digits 8 and 9 both require four bits, but the rest require only three. So six binary strings are not used and cannot represent digits. For example, $1010_2 = 10 > 9$, so 1010_2 will never appear in a correct BCD digit encoding. In BCD, results mostly are computed digit-by-digit in binary and then manipulated into a correct BCD encoding.

Using four bits per digit has one major advantage; each decimal digit is a hexadecimal digit. So the hex number 1594_{16} is interpreted as the decimal 1594.

This also allows a nifty trick for adding two BCD-encoded numbers.

Say we want to add $a = 1103$ and $b = 328$. In decimal, $a + b = 1431$. If we were to add these directly in hexadecimal, $a + b = 142\mathsf{B}_{16}$. There needs to be some mechanism for carrying. We can use the six missing code points to force a carry into the next digit, and then we can compare with the *exclusive-or* to detect where carries actually happened.

The procedure starts by adding 6 to each BCD digit as if they were hexadecimal. So we shift each digit of a to the top of its hex range and use $a+6666_{16} = 7769_{16}$. Now we compute a sum $s_1 = (a+6666_{16})+b = 7\mathsf{A}91_{16}$. This isn't the final sum; if we subtract 6 from every digit, $7\mathsf{A}91_{16} - 6666_{16} = 142\mathsf{B}_{16}$, we do not obtain a BCD-encoded number.

We need to subtract 6 only from those digits that did not generate a carry, $7\mathsf{A}91_{16} - 6660_{16} = 1431_{16}$. This is a correct BCD number and the correct result. The carries can be detected by comparing $(a+6666_{16})+b$ with $(a+6666_{16}) \oplus b$, the bitwise *exclusive or*. If the two results differ in the lowest bit per hex digit / BCD digit, we know there was a carry and we know where to subtract 6_{16}.

Alas, there are no particularly nice tricks for multiplication. But if most uses include adding a list of prices and applying a tax once, it's not so bad.

Another form that wastes far less space is called **millennial encoding**. Because $2^{10} = 1024 > 10^3$, ten bits can represent all three decimal digit numbers. This wastes only 25 encoding points per three decimal digits, as opposed to wasting six points every for every single decimal digit. Arithmetic operates in binary on the chunks of ten bits and then manipulates the results.

And there are more encodings, including Tien Chi Chen and Dr. Irving T. Ho's **Chen-Ho** encoding (1975), Mike Cowlishaw's **DPD** encoding (**d**ensely **p**acked **d**ecimal, 2002), and Intel's **BID** encoding (**b**inary **i**nteger **d**ecimal). These require more complicated coding techniques to explain, but the latter two (DPD and BID) are now (as of August, 2008) international standards.

Chapter 22

Homework for the sixth week: numbers and computing

22.1 Homework

Practice is absolutely critical in this class.

Groups are fine, turn in your own work. Homework is due in or before class on Mondays.

- Section 4.1:
  - Problems 35, 36 (the algorithm is in the text, see Section 4.1, Example 4)
- Section 4.2:
  - Problems 2, 3, 5, 6, 11, 12
- Section 4.3:
  - Problem 2, 7, 8
  - Problems 19-22 (the "calculator shortcut" is Horner's rule)
  - Problems 37-40
  - Problem 57 (he played at the festival)
- Expressing numbers in positional form:

  - Take a familiar incomplete integer, _679_, and express it as a sum of the digits times powers of ten using variables x_0 and x_4 for the digits in the blanks. Simplify to the form of $x_4 \cdot 10^4 + x_0 \cdot 10^0 + z$, where z is a single number in positional form (a sequence of digits). Does 72 divide z? Does 8 divide z? Does 9 divide z? Remember that $72 = 8 \cdot 9$. We will use this example again in the next chapter.

- Operations;
  - Multiply 47 by each of 3, 13, and 23. Show your work, and work digit-by-digit. Use either the expanded form (expanding $(4 \cdot 10+7) \cdot (2 \cdot 10+3)$ or the tabular form collapsing the sum every two steps.
  - Add 47 to each of 52, 53, and 54. Show your work, and work digit-by-digit. Show an intermediate redundant representation if there is one.
  - Subtract 19 from each of 7, 19 (not a typo), 20, and 29. Show your work, and work digit-by-digit. Show an intermediate redundant representation if there is one.

Note that you *may* email homework. However, I don't use Microsoft$^{\text{TM}}$ products (*e.g.* Word), and software packages are notoriously finicky about translating mathematics.

If you're typing it (which I advise just for practice in whatever tools you use), you likely want to turn in a printout. If you do want to email your submission, please produce a PDF or PostScript document.

Chapter 23

Solutions for sixth week's assignments

Also available as PDF.

23.1 Section 4.1, problems 35 and 36

Multiplying $26 \cdot 53$ by the Egyptian algorithm:

1	53
2	106
4	212
8	424
16	848

Now $26 = 16 + 8 + 2$, so $26 \cdot 53 = 848 + 424 + 106 = \mathbf{1378}$.

Computing $33 \cdot 81$:

1	81
2	162
4	324
8	648
16	1296
32	2592

Because $33 = 32 + 1$, $33 \cdot 81 = 2592 + 81 = \mathbf{2673}$.

23.2 Section 4.2

Problem 2 $925 = 9 \cdot 10^2 + 2 \cdot 10^1 + 5 \cdot 10^0$

Problem 3 $3774 = 3 \cdot 10^3 + 7 \cdot 10^2 + 7 \cdot 10^1 + 4 \cdot 10^0$

Problem 5 $4 \cdot 10^3 + 9 \cdot 10^2 + 2 \cdot 10^1 + 4 \cdot 10^0$

Problem 6 $5 \cdot 10^4 + 2 \cdot 10^3 + 1 \cdot 10^2 + 1 \cdot 10^1 + 8 \cdot 10^0$

Problem 11 6209

Problem 12 503568

23.3 Section 4.3

Problem 2 1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 20, 21, 22, 23, 24

Problem 7 $\mathsf{B}\mathbf{6E}_{\mathbf{16}}$, $\mathsf{B6F}_{16}$, $\mathsf{B}\mathbf{70}_{\mathbf{16}}$

Problem 8 $\mathbf{10110_2}$, 10111_2, $\mathbf{11000_2}$

Problem 19 $\mathsf{3BC}_{16} = (3 \cdot 16 + 11) \cdot 16 + 12 = 956$

Problem 20 $34432_5 = (((3\dot{5} + 4) \cdot 5 + 4) \cdot 5 + 3) \cdot 5 + 2 = 2492$

Problem 21 $2366_7 = ((2 \cdot 7 + 3) \cdot 7 + 6) \cdot 7 + 6 = 881$

Problem 22 $101101110_2 = (((((((1 \cdot 2 + 0) \cdot 2 + 1) \cdot 2 + 1) \cdot 2 + 0) \cdot 2 + 1) \cdot 2 + 1) \cdot 2 + 1) \cdot 2 + 0 = 366$

Problem 37 $586 = 512 + 64 + 8 + 2 = 2^9 + 2^6 + 2^3 + 2^1 = 1001001010_2$

Problem 38 $12888 = 3 \cdot 4096 + 1 \cdot 512 + 1 \cdot 64 + 3 \cdot 8 = 3 \cdot 8^4 + 1 \cdot 8^3 + 1 \cdot 8^2 + 3 \cdot 8 = 31130_8$

Problem 39 $8407 = 102112101_3$

Problem 40 $11028 = 2230110_4$

Problem 57 $9 \cdot 12^2 + 10 \cdot 12 + 11 = 1427$

23.4 Positional form

> Take a familiar incomplete integer, _679_, and express it as a sum of the digits times powers of ten using variables x_0 and x_4 for the digits in the blanks. Simplify to the form of $x_4 \cdot 10^4 + x_0 \cdot 10^0 + z$, where z is a single number in positional form (a sequence of digits). Does 72 divide z? Does 8 divide z? Does 9 divide z? Remember that $72 = 8 \cdot 9$. We will use this example again in the next chapter.

We can expand _ 679 _ to be $x_4 \cdot 10^4 + 6 \cdot 10^3 + 7 \cdot 10^2 + 9 \cdot 10 + x_0 = x_4 \cdot 10^4 + x_0 + 6790$, so $\mathbf{z} = \mathbf{6790}$. Unfortunately, none of the numbers provided divide cleanly into 6790. Jumping to the next chapter, $6790 = 94 \cdot 72 + 22 = 848 \cdot 8 + 6 = 754 \cdot 9 + 4$.

23.5 Operations

Multiplication:

- $47 \cdot 3 = (4 \cdot 10 + 7) \cdot 3 = 12 \cdot 10 + 21 = 141$.
  Or in table form:

$$\begin{array}{cr}
 & 47 \\
\cdot & 3 \\
\hline
 & 21 \\
 & 12 \\
\hline
 & 141
\end{array}$$

- $47 \cdot 13 = (4 \cdot 10 + 7) \cdot (10 + 3) = (4 \cdot 10^2 + 7 \cdot 10) + (12 \cdot 10 + 21) = 470 + 141 = 611$.
  Or in table form:

$$\begin{array}{cr}
 & 47 \\
\cdot & 13 \\
\hline
 & 21 \\
 & 12 \\
\hline
 & 141 \\
 & 47 \\
\hline
 & 611
\end{array}$$

- $47 \cdot 23 = (4 \cdot 10 + 7) \cdot (2 \cdot 10 + 3) = (8 \cdot 10^2 + 14 \cdot 10) + (12 \cdot 10 + 21) = 940 + 141 = 1081$.
  Or in table form:

$$\begin{array}{cr}
 & 47 \\
\cdot & 23 \\
\hline
 & 21 \\
 & 12 \\
\hline
 & 141 \\
 & 14 \\
\hline
 & 240 \\
 & 8 \\
\hline
 & 1081
\end{array}$$

Addition:

- $47 + 52 = (4 + 5) \cdot 10 + (7 + 2) = 9 \cdot 10 + 9 = 99$.
- $47+53 = (4+5)\cdot 10+(7+3) = \mathbf{9 \cdot 10 + 10} = \mathbf{10 \cdot 10 + 0} = 1\cdot 10^2+0\cdot 10+0 = 100$. Both bold forms are redundant intermediate representations.
- $47+54 = (4+5)\cdot 10+(7+4) = \mathbf{9 \cdot 10 + 11} = \mathbf{10 \cdot 10 + 1} = 1\cdot 10^2+0\cdot 10+1 = 101$. Both bold forms are redundant intermediate representations.

Subtraction:

- $7 - 19 = (0 - 1) \cdot 10 + (7 - 9) = -\mathbf{1} \cdot \mathbf{10} + -\mathbf{2}\cdot = -1 \cdot (1 \cdot 10 + 2) = -12$. The bold form is a redundant intermediate representation.
- $19 - 19 = (1 - 1) \cdot 10 + (9 - 9) = 0 + 0 = 0$.
- $20 - 19 = (2 - 1) \cdot 10 + (0 - 9) = \mathbf{1 \cdot 10} + -\mathbf{9} = 0 \cdot 10 + (10 - 9) = 1$. The bold form is a redundant intermediate representation.
- $29 - 19 = (2 - 1) \cdot 10 + (9 - 9) = 1 \cdot 10 + 0 = 10$.

Chapter 24

Notes for the seventh week: primes, factorization, and modular arithmetic

Notes also available as PDF.

What we will cover from Chapter 5:

- divisibility and prime numbers,
- factorization into primes,
- modular arithmetic,
- finding divisibility rules,
- greatest common divisors and least common factors,
- Euclid's algorithm for greatest common divisors, and
- solving linear Diophantine equations.

Once upon a time, number theory was both decried and revered as being "pure mathematics" with no practical applications. That is no longer remotely true. There are oblique applications in error correction (*e.g.* how CDs still play when scratched), but one overwhelming, direct application is in **encryption**. So I also will discuss

- Euler's totient function ($\phi(n)$) and the RSA encryption algorithm. **Alas, we didn't reach this. We might cover it in the future.**

The RSA algorithm is at the core of the ***s**ecure **s**ocket **l**ayer* (SSL) protocol used to secure web access (the `https` prefix, colored locks, *etc.*).

This week, we will cover divisibility, primes, factorization, and modular arithmetic.

24.1 Divisibility

When defining operations on integers, we skipped division. As with subtraction, the integers are not closed over division; 1/2 is not an integer. So we define division implicitly.

For any integers a and b, we can write

$$b = q \cdot a + r,$$

where q is an integer called the **quotient**, and $r < |a|$ is a *non-negative* integer called is the **remainder**, **residue**, or **residual**. We will see that requiring $0 \leq r < |a|$ is very important and makes division well-defined.

Then a **divides** b, or $a \mid b$, when $r = 0$. Alternately, b is a **multiple** of a and a is a **divisor** of b. If we cannot write $b = q \cdot a + r$ with $r = 0$, then a does not divide b, or $a \nmid b$. When $a \mid b$, then we define division as $b/a = q$.

For example,

$$\begin{aligned}
14 &= 2 \cdot 7 + 0, \text{ so } 7 \mid 14 \text{ and } 14/7 = 2, \text{ and} \\
20 &= 2 \cdot 7 + 6, \text{ so } 7 \nmid 20 \text{ and } 20/7 \text{ is not an integer.}
\end{aligned}$$

In the latter case, though, $20/7 = 2 + 6/7$, which rounds down to 2.

Some other examples showing extreme and negative cases,

$$\begin{aligned}
-6 &= -3 \cdot 2 + 0, \text{ so } 2 \mid -6 \text{ and } -6/2 = -3, \\
-6 &= 3 \cdot -2 + 0, \text{ so } -2 \mid -6 \text{ and } -6/-2 = 3, \\
6 &= -3 \cdot -2 + 0, \text{ so } -2 \mid 6 \text{ and } 6/-2 = -3, \\
-7 &= -4 \cdot 2 + 1, \text{ so } 2 \nmid -7, \\
-7 &= 4 \cdot -2 + 1, \text{ so } -2 \nmid -7, \\
7 &= -3 \cdot -2 + 1, \text{ so } -2 \nmid 7 \text{ (}\textbf{note}\text{: not } -4 \cdot -2 - 1\text{)}, \\
5 &= 0 \cdot 10 + 5, \text{ so } 10 \nmid 5, \text{ and} \\
0 &= 0 \cdot 13 + 0, \text{ so } 13 \mid 0 \text{ and } 0/13 = 0.
\end{aligned}$$

What about when $a = 0$? Then $b = q \cdot 0 + b$ is true for any quotient q. Without further restrictions on q, division by zero is not be well-defined. In calculus and some applications, there are times when you fill in the hole left by a division by zero by some obvious completion.

But is the form $b = qa + r$ well-defined when $a \neq 0$?

Theorem: The expansion $b = qa + r$ with $0 \leq r < |a|$ is unique for $a \neq 0$, so division is well-defined.

Proof. We begin by assuming there are two ways of expanding $b = qa + r$. Then we show that the forms must be identical.

Let there be two distinct ways of writing $b = qa + r$ with $a \neq 0$,

$$b = q_1 a + r_1, \text{ and}$$
$$b = q_2 a + r_2,$$

with $0 \leq r_1 < |a|$ and $0 \leq r_2 < |a|$.

If $r_1 = r_2$, then $b - r_1 = b - r_2$. From the equations above $b - r_1 = q_1 a$ and $b - r_2 = q_2 a$, so $q_1 a = q_2 a$ or $(q_1 - q_2)a = 0$. Because $a \neq 0$, $q_1 = q_2$ and the forms are identicall.

For $r_1 \neq r_2$, we know one of them is larger. *Without loss of generality*, assume $r_1 < r_2$. Then there is some positive integer k such that increases r_1 to match r_2, or $r_2 = r_1 + k$. Note that $k \leq r_2$.

Substituting for r_2, we see that $b = q_2 a + r_1 + k$, or equivalently $b - k = q_2 a + r_1$. Now we can subtract this equation from $b = q_1 a + r_1$ to obtain

$$k = (q_1 - q_2)a = z \cdot a + 0$$

for some quotient z.

So $a \mid k$, but $k \leq r_2 < |a|$. The only way we can satisfy this is if $q_1 - q_2 = 0$ and $q_1 = q_2$. Thus also $k = 0$ and $r_1 = r_2$. So we cannot have to different ways to write $b = q \cdot a + r$, and our form of division is well-defined. □

Some useful properties of divisibility:

- If $d \mid a$ and $d \mid b$, then $d \mid ra + sb$ for all integers r and s. A quick proof: $a = q_a d$ and $b = q_b d$, so $ra + sb = rq_a d + sq_b d = (rq_a + sq_b)d$, then $d | ra + sb$.
- If $a \mid b$ and $b \mid c$, then $a \mid c$. Quick proof: $b = q_a a$, $c = q_b b$, so $c = q_b(q_a a) = (q_b q_a)a$.
- If $a \mid bc$ and $a \nmid b$, then $a \mid c$.

24.2 Primes

Divisibility gives a numbers a multiplicative structure that's different than the digit-wise structure we previously examined.

To build the structure, we start from numbers which cannot be decomposed. An integer $p > 1$ is called a **prime** number if its only divisors are 1 and p itself. We will explain why 1 is not considered prime when we discuss factorization. All other numbers are **composite** and must have some prime divisor.

Consider possible *divisors* of 11,

$$\begin{aligned} 11 &= 11 \cdot 1 + 0 \text{ so } 1 \mid 11, \\ 11 &= 5 \cdot 2 + 1 \text{ so } 2 \nmid 11, \text{ and} \\ 11 &= 3 \cdot 3 + 2 \text{ so } 3 \nmid 11. \end{aligned}$$

We can stop at 3. Because multiplication is commutative, any divisors come in pairs. The smaller of the pair must be $\leq \sqrt{11} \approx 3.3$; that's the point where any pairs $a \cdot b$ are repeated as $b \cdot a$.

So the only divisor less than $\sqrt{11}$ is 1, and 11 is prime.

How many primes are there?
Theorem: There are infinitely many primes.

Proof. Assume there are only k primes $p_1, p_2, \ldots, p_k$ and all other numbers are composite. Then let $n = p_1 \cdot p_2 \cdot \cdots \cdot p_k + 1$, one larger than the product of all primes.

Consider dividing n by some prime, say p_k. Then we can write $n = (p_1 \cdot p_2 \cdot \cdots \cdot p_{k-1})p_i + 1$. Given the form is unique and $r = 1$, p_k does not divide n. We could have chosen any of the primes, so $p_i \nmid n$ for all $i = 1, \ldots, k$. Thus no prime divides n.

For a number n to be composite, it must have some factor or divisor other than 1 and n. If that factor is not prime, then the factor has another factor, and so forth until you reach some prime. Because of transitivity of division ($a \mid b$ and $b \mid c$ imply $a \mid c$), the prime must divide n. Here, though, no primes divide n, so n cannot be composite and must be prime itself.

So assuming there are k primes leads to a contradiction because we can construct one more. Thus there are either no primes or infinitely many. We demonstrated that 11 is prime, so there must be infinitely many primes. □

There are mountains of unanswered questions about prime numbers. Consider the pairs of primes $(3, 5)$, $(11, 13)$, and $(17, 19)$. Each are separated by two. Are there infinitely many such pairs? No one knows. Similarly, there are **Mersenne primes** of the form $2^n - 1$. No one knows how many Mersenne primes exist.

24.3 Factorization

A **factorization** of a number is a decomposition into factors. So $24 = 8 \cdot 3$ is a factorization of 24, as is $24 = 4 \cdot 2 \cdot 3$. A **prime factorization** is a factorization into primes. Here $24 = 2 \cdot 2 \cdot 2 \cdot 3$ is a prime factorization of 24. We use exponents to make this easier to write, and $24 = 2^3 \cdot 3$.

You can be systematic about prime factorization and discover the primes through the **sieve of Eratosthenes**. Consider finding a prime factorization of 1100.

We start just by writing possible factors. Technically we need integers only $\leq \sqrt{1100} \approx 33.6$, but we only fill enough here to demonstrate the point.

	2	**3**	*4*	**5**	*6*	**7**	*8*	*9*	*10*
11	*12*	**13**	*14*	*15*	*16*	**17**	*18*	**19**	*20*

The first possible factor is 2, and $2 \mid 1100$. We divide by two until the result is not divisible by 2. This gives $1100 = 2^2 \cdot 275$. Then we cross out all multiples of 2; these cannot divide 275. Because $275 = 91 \cdot 3 + 2$, $3 \nmid 275$. But we also know no multiples of 3 divide 275, so we cross out all remaining multiples of 3.

The next not crossed out is 5, which divides 275. Now $1100 = 2^2 \cdot 5^2 \cdot 11$. We showed 11 is prime before, but let's continue this method. Cross out all multiples of five. The next number to try is 7, which does not divide 11. But we can cross out all multiples of 7. The next free number is 11, which we have again shown to be prime.

In our (short) list, it happens that only primes remain. We have sieved out all the non-primes. Actually, once we removed all multiples of primes $\leq \sqrt{20}$, only primes remained.

Moving from one prime to the next is a systematic method both for finding prime numbers and for finding a prime factorization.

Factorizations provide a useful mechanism for working

We will not prove the following, but is often called the **fundamental theorem of arithmetic**:
Theorem: Every integer greater than one has a unique prime factorization.

This theorem, which we will not prove, is why 1 is not considered a prime. If it were, $10 = 2 \cdot 5 = 1 \cdot 2 \cdot 5 = 1^2 \cdot 2 \cdot 5 = \ldots$ would all be prime factorizations of 10.

24.4 Modular Arithmetic

Factorization itself will prove useful later. Now we will explore modular arithmetic and find some quick rules for determining when $d \mid a$ for some d.

Modular arithmetic is arithmetic on remainders.

Consider expressions of 7 and 4 in terms of multiples of 3 plus remainders: $7 = 2 \cdot 3 + 1$ and $4 = 1 \cdot 3 + 1$. Now $11 = 7 + 4 = (2 \cdot 3 + 1) + (1 \cdot 3) + 1 = 3 \cdot 3 + 2$. Note that the sum of the remainders was < 3 and was the new remainder itself.

If the remainder is ≥ 3, we can just pull a three out of it: $11 + 8 = (3 \cdot 3 + 2) + (2 \cdot 3 + 2) = 5 \cdot 3 + 4$. To convert this into the correct form, note that $4 = 1 \cdot 3 + 1$, and $19 = 5 \cdot 3 + (1 \cdot 3 + 1) = 6 \cdot 3 + 1$. We need consider only the sum of remainders to compute the result's remainder.

The remainder of the sum just wraps around. Think about time. If you add a few hours and cross 12, the result just wraps around. So 1:00 is the same as 13:00 or 25:00.

We don't identify 1:00 as just one time but a member of a set of all times that are one hour after a multiple of 12. Similarly, we can identify numbers as elements of sets where all members have the same remainder relative to a given divisor.

The **congruence class** of r **modulo** a is $\{x \mid \exists q : x = qa + r\}$. If a number b is in the congruence class of r modulo a, we write $b \equiv r \pmod{a}$.

The canonical member of a congruence class is its least positive member. Just as we don't naturally consider 25:00 as 1:00, we tend to identify congruence classes by the least r. So while $13 \equiv 87 \pmod{2}$ is correct (both 13 and 87 are odd), we prefer $13 \equiv 1 \pmod{2}$.

We define addition and multiplication on entire congruence classes. For the operation to be defined, the **modulus** of each class must be the same. Then we're adding numbers of the form $b_1 = q_1 a + r_1$ for $b_1 \equiv r_1 \pmod{a}$ and $b_2 = q_2 a + r_2$ for $b_2 \equiv r_2 \pmod{a}$. As in our example above, the remainders add. Here $b_1 + b_2 = q_1 a + r_1 + q_2 a + r_2 = (q_1 + q_2)a + (r_1 + r_2) \equiv r_1 + r_2 \pmod{a}$.

Identifying congruence classes by their least positive element, we can write a table showing all additions modulo 4:

+ (mod 4)	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Addition of congruence classes maintains the **additive identity** that we expect, $b + 0 \equiv b \pmod{a}$.

Note that every class has an **additive inverse**, a class where $b + (-b) \equiv 0 \pmod{a}$. Remember that we forced the residual to be positive when we defined division. Then we can see that the inverse of 1 modulo 4 is $-1 = -1 \cdot 4 + 3 \equiv 3 \pmod{4}$.

Another way to see this is that the canonical representation of $-b$ is the least number which increases b to be equal to the modulus a. So the inverse of 1 is 3 because $1 + 3 = 4 \equiv 0 \pmod{4}$.

We also define multiplication on congruence classes.

If $b_1 = q_1 a + r_1$ and $b_2 = q_2 a + r_2$, then

$$\begin{aligned} b_1 \cdot b_2 &= (q_1 a + r_1) \cdot (q_2 a + r_2) \\ &= q_1 q_2 a^2 + q_1 r_2 a + q_2 r_1 a + r_1 r_2 \\ &= (q_1 q_2 a + q_1 r_2 + q_2 r_1) a + r_1 r_2 \\ &\equiv r_1 r_2 \pmod{a}. \end{aligned}$$

So we need only multiply remainders.

Identifying congruence classes by their least positive element, we can write a table showing all multiplications modulo 4:

$\times \pmod{4}$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Again, there is a **multiplicative identity**, $b \cdot 1 \equiv b \pmod{a}$.

Unlike plain integer division, some congruence classes have an inverse. The only integer that has an integer inverse is 1. But modulo 4, both 1 and 3 have **multiplicative inverses**. Here $3 \cdot 3 = 9 \equiv 1 \pmod{3}$.

24.5 Divisibility Rules

Using modular arithmetic and positional notation, we can derive some quick divisibility tests.

First, consider divisibility by powers of 2 and 5. The factorization of $10 = 2 \cdot 5$, and so $10^k = 2^k \cdot 5^k$. So $2^k \mid 10^k$ and $5^k \mid 10^k$, or $10^k \equiv 0 \pmod{2}^k$ and $10^k \equiv 0 \pmod{5}^k$.

Now remember how to expand positional notation. We know that $1234 = 1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10^1 + 4$. So $1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10^1 + 4 \equiv 0 + 0 + 0 + 4 \pmod{2} \equiv 0 \pmod{2}$. Divisibility by 2 depends only on the final digit. Similarly, $1234 \equiv 0 + 0 + 0 + 4 \pmod{5}$, and divisibility by 5 depends only on the final digit.

For $2^2 = 4$ and $5^2 = 25$, all but the last two digits are equivalent to zero. And for $2^3 = 8$ and $5^3 = 125$, all but the last three digits are equivalent to zero. So one divisibility rule:

> When testing for divisibility by 2^k or 5^k, we need only consider the last k digits.

Now consider divisibility by 3 or 9. We know that $10 \equiv 1 \pmod 3$ and $10 \equiv 1 \pmod 9$. Using modular arithmetic, $123 = 1 \cdot 10^2 + 2 \cdot 10 + 3 \equiv 1 + 2 + 3 \pmod 3 \equiv 6 \pmod 3 \equiv 0 \pmod 3$. Hence $3 \mid 123$ because the sum of its digits is divisible by 3.

Similarly, $10 \equiv 1 \pmod 3$. So $123 \equiv 1+2+3 \pmod 9 \equiv 6 \pmod 9$, and $9 \nmid 123$. If the sum of the digits is greater than 9, simply add those digits.

> Test for divisibility by 3 or 9 by adding the number's digits and checking that sum. If that sum is greater than 9, add the digits again. Repeat until the result is obvious.

Other primes are not so straight-forward. Divisibility by 7 is a pain; there is an example method in the text's problems for Section 5.1.

The rule for 11 is worth exploring. Because $10 < 11$, the canonical member of its congruence class is just 10. But there is another member of interest, $10 \equiv -1 \pmod 1$1. So you can alternate signs on alternate digits from the right. So $123456 \equiv -1 + 2 - 3 + 4 - 5 + 6 \equiv 3 \pmod 1$1, and $11 \nmid 123456$.

For divisibility by 6, 12, 18, or other composite numbers, factor the divisor and test for divisibility by each factor. To test for divisibility by $72 = 2^3 \cdot 3^2 = 8 \cdot 9$, test for divisibility by 8 and by 9.

Chapter 25

Homework for the seventh week: primes, factorization, and modular arithmetic

25.1 Homework

Notes also available as PDF.

Practice is absolutely critical in this class.

Groups are fine, turn in your own work. Homework is due in or before class on Mondays.

- Section 5.1 (prime numbers):
  - 3, 4, 5, 7
  - 14, 15, 16
  - 80
- Section 5.1 (factorization):
  - 34-36
  - 56-59
- Section 5.4 (modular arithmetic):
  - 9-13 (this is modulo 5, and the inverse of a is a number b such that $a + b \equiv 0 \pmod 5$)
  - 29, 31

  - 33, 35, 37, 39
- Section 5.1 (divisibility rules):
  - 21-24
  - 43-44
  - Take a familiar incomplete integer, _679_. Using the expression of _679_ as $N = 10^4 \cdot x_4 + x_0 + 6790$, use $8 \mid N$ to find x_0? Given that, use $9 \mid N$ to find x_4. Now if 72 turkeys cost \$_679_, what is the total?

Note that you *may* email homework. However, I don't use Microsoft™ products (*e.g.* Word), and software packages are notoriously finicky about translating mathematics.

If you're typing it (which I advise just for practice in whatever tools you use), you likely want to turn in a printout. If you do want to email your submission, please produce a PDF or PostScript document.

Chapter 26

Solutions for seventh week's assignments

Also available as PDF.

26.1 Section 5.1 (prime numbers)

Problem 3 There is **one** even prime number, 2. The statement is **false**.

Problem 4 If $9 \mid n$, then $n = 9 \cdot k$ for some integer k. Then also $n = 3 \cdot (3 \cdot k) = 3 \cdot k'$ for an integer $k' = 3k$, so $3 \mid n$. The statement is **true**.

Problem 5 Here, $5 \mid 15$ but $10 \nmid 15$, so the statement is **false**.

Problem 7 A number $n = 1 \cdot n$. This is of the divisibility form $n = 1 \cdot n + 0$ with a remainder of zero, so $n \mid n$ and n is a factor of itself. Also $n \cdot 1 = n$ and n is a multiple of itself. Thus the statement is **true**.

Problem 14 $18 = 1 \cdot 18 = 2 \cdot 9 = 3 \cdot 6$, so its factors are 1, 2, 3, 6, 9, and 18.

Problem 15 $20 = 1 \cdot 20 = 2 \cdot 10 = 4 \cdot 5$, so its factors are 1, 2, 4, 5, 10, and 20.

Problem 16 $28 = 1 \cdot 28 = 2 \cdot 14 = 4 \cdot 7$, so its factors are 1, 2, 4, 7, 14, and 28.

26.1.1 Problem 80

First evaluate $M = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30031$. To check if this is a prime or composite, we need to check for factors up to $\sqrt{30031} \approx 173.3$ and hence up to 173.

Using Eratosthenes's sieve method to search through integers:

	2	**3**	$\cancel{4}$	**5**	$\cancel{6}$	**7**	$\cancel{8}$	**9**	$\cancel{10}$
11	$\cancel{12}$	**13**	$\cancel{14}$	$\cancel{15}$	$\cancel{16}$	**17**	$\cancel{18}$	**19**	$\cancel{20}$
$\cancel{21}$	$\cancel{22}$	**23**	$\cancel{24}$	$\cancel{25}$	$\cancel{26}$	$\cancel{27}$	$\cancel{28}$	**29**	$\cancel{30}$
31	$\cancel{32}$	$\cancel{33}$	$\cancel{34}$	$\cancel{35}$	$\cancel{36}$	**37**	$\cancel{38}$	$\cancel{39}$	$\cancel{40}$
41	$\cancel{42}$	**43**	$\cancel{44}$	$\cancel{45}$	$\cancel{46}$	**47**	$\cancel{48}$	$\cancel{49}$	$\cancel{50}$
$\cancel{51}$	$\cancel{52}$	**53**	$\cancel{54}$	$\cancel{55}$	$\cancel{56}$	$\cancel{57}$	$\cancel{58}$	$\boxed{\mathbf{59}}$	

Once we reach the prime 59, we find $59 \mid 30031$ and thus **30031 is not a prime**. Its factorization is $\mathbf{59 \cdot 509}$.

26.2 Section 5.1 (factorization)

Problem 34 $425 = 5^2 \cdot 17$

Problem 35 $663 = 3 \cdot 13 \cdot 17$

Problem 36 $885 = 3 \cdot 5 \cdot 59$

The above could be solved by any reasonable method. As an example of the sieve method, consider the last problem of factoring 885.

	2	$\boxed{\mathbf{3}}$	$\cancel{4}$	$\boxed{\mathbf{5}}$
$\cancel{6}$	**7**	$\cancel{8}$	$\cancel{9}$	$\cancel{10}$

We start to list numbers up to $\sqrt{885} \approx 29.7$ but only list them one line at a time. For the first prime, $2 \nmid 885$. Then $3 \mid 885$ and $885/3 = 295$. Only one factor of 3 can be pulled out, so $885 = 3 \cdot 295$. Now we only need to check numbers up to $\sqrt{295} \approx 17.2$. The next prime $5 \mid 295$, and $295/5 = 59$. Because $5 \nmid 59$, only one factor of five can be pulled out. Now $885 = 3 \cdot 5 \cdot 59$.

A previous problem showed that 59 is a prime, so we could stop here. Or if we didn't know that 59 is prime, we need to check for factors up to $\sqrt{59} \approx 7.7$. Only one more prime, 7, is in that range, and $7 \nmid 59$, so 59 is prime.

The prime factorization is $\mathbf{885 = 3 \cdot 5 \cdot 59}$.

Problem 56 $48 = 2^4 \cdot 3^1$, so there are $(4+1) \cdot (1+1) = \mathbf{10}$ factors.

Problem 57 $72 = 2^3 \cdot 3^2$, so there are $(3+1) \cdot (2+1) = \mathbf{12}$ factors.

Problem 58 $144 = 12^2 = (2^2 \cdot 3^1)^2 = 2^4 \cdot 3^2$, so there are $(4+1) \cdot (2+1) = \mathbf{15}$ factors.

Problem 59 $2^8 \cdot 3^2 = 2304$, and there are $(8+1) \cdot (2+1) = \mathbf{27}$ factors.

26.3 Section 5.4 (modular arithmetic)

26.3.1 Problems 9-13

The operation table is

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Every entry is an integer no less than zero and less than five, so the operation as defined is **closed**. Considering congruence classes rather than clocks, every integer is in **one** congruence class, so operations that produce integers must be closed.

The table is symmetric along its diagonal $(0 \cdot 0,\ 1 \cdot 1,\ \ldots)$, so the operation is **commutative**. Adding $0 + a \equiv a \pmod 5$, so **zero is the additive identity**. And adding $a + (5 - a) \equiv 0 \pmod 5$, so $\mathbf{5 - a}$ **is the additive inverse**.

26.3.2 Other problems

Problem 29 $19 = 6 \cdot 3 + 1$ and $5 = 1 \cdot 3 + 2$. So $\mathbf{19} \not\equiv \mathbf{5} \pmod{\mathbf{3}}$.

Problem 31 The sum of the digits is $9 + 9 \equiv 0 \pmod 3$, so $3 \mid 5445$ and $\mathbf{5445} \equiv \mathbf{0} \pmod{\mathbf{3}}$.

Problem 33 $12 + 7 = 19 \equiv \mathbf{3} \pmod{\mathbf{4}}$.

Problem 35 $35 - 22 = 13 \equiv \mathbf{3} \pmod{\mathbf{5}}$.

Problem 37 $5 \cdot 8 \equiv -1 \cdot -1 \equiv \mathbf{1} \pmod{\mathbf{3}}$.

Problem 39 $4 \cdot (13 + 6) = 4 \cdot 19 \equiv 4 \cdot 8 \equiv 4 \cdot (-3) \equiv -12 \equiv -1 \equiv \mathbf{10} \pmod{\mathbf{11}}$.

26.4 Section 5.1 (divisibility rules)

These can be found either with the rules in the text or by evaluating each number:

Problem 21 $5 \mid 25025$

Problem 22 $5 \mid 45815$

Problem 23 $3 \mid 123\,456\,789$, $9 \mid 123\,456\,789$

Problem 24 $3 \mid 987\,654\,321$, $9 \mid 987\,654\,321$ (same digits, different order)

Using the method described in class:

Problem 43 $453\,896\,248 \equiv 8-4+2-6+9-8+3-5+4 \equiv 3 \pmod{11}$, so $11 \nmid 453\,896\,248$.

Problem 44 $522\,749\,913 \equiv 3-1+9-9+4-7+2-2+5 \equiv 4 \pmod{11}$, and $11 \nmid 522\,749\,913$.

26.4.1 Take a familiar incomplete integer...

> Take a familiar incomplete integer, _679_. Using the expression of _679_ as $N = 10^4 \cdot x_4 + x_0 + 6790$, use $8 \mid N$ to find x_0. Given that, use $9 \mid N$ to find x_4. Now if 72 turkeys cost \$_679_, what is the total?

If $8 \mid N$, then we need only check the last three digits. So $790 + x_0 = 8 \cdot k$ for some k. There are only four values of x_0 worth checking: 0, 2, 4, and 8. Of these, only $8 \mid 792$, so $x_0 = 2$.

Now $9 \mid N$ implies $9 \mid x_4 \cdot 10^4 + 6792$. Adding just the digits suffices, so we must solve $x_4 + 6 + 7 + 9 + 2 \equiv 0 \pmod{9}$. Hence $x_4 + 6 \equiv 0 \pmod{9}$, and $x_4 \equiv -6 \equiv 3 \pmod{9}$. The only such x_4 that satisfies $0 \leq x_4 \leq 9$ is $x_4 = 3$.

Thus $\mathbf{x_4 = 3}$, $\mathbf{x_0 = 2}$ and the total price is **\$367.92**. The price per turkey is \$5.11.

Chapter 27

Notes for the eighth week: GCD, LCM, $ax + by = c$

Notes also available as PDF.

What we covered last week:

- divisibility and prime numbers,
- factorization into primes,
- modular arithmetic,
- finding divisibility rules,

This week's topics:

- review modular arithmetic and finding divisibility rules,
- greatest common divisors and least common factors,
- Euclid's algorithm for greatest common divisors, and
- solving linear Diophantine equations.

These all are useful when you deal with integral numbers of things

27.1 Modular arithmetic

Remember the divisibility form for b with respect to dividing by $a \neq 0$,

$$b = q \cdot a + r, \text{ with } 0 \leq r < |a|.$$

This form is *unique* for a given a and b.

Consider $a = 5$. There are only five possible values of r, zero through four. Because the form is unique, we can place every b into one of r **congruence classes**. Each congruence class is a set. For $a = 5$, we have the following classes:

$$
\begin{array}{lrrrrrll}
\{\ldots, & \text{-10}, & \text{-5}, & \mathbf{0}, & 5, & 10, & \ldots\} & = \{5k+0 \mid k \in \mathbb{J}\} \\
\{\ldots, & \text{-9}, & \text{-4}, & \mathbf{1}, & 6, & 11, & \ldots\} & = \{5k+1 \mid k \in \mathbb{J}\} \\
\{\ldots, & \text{-8}, & \text{-3}, & \mathbf{2}, & 7, & 12, & \ldots\} & = \{5k+2 \mid k \in \mathbb{J}\} \\
\{\ldots, & \text{-7}, & \text{-2}, & \mathbf{3}, & 8, & 13, & \ldots\} & = \{5k+3 \mid k \in \mathbb{J}\} \\
\{\ldots, & \text{-6}, & \text{-1}, & \mathbf{4}, & 9, & 14, & \ldots\} & = \{5k+4 \mid k \in \mathbb{J}\}
\end{array}
$$

We say that two numbers are in the same congruence class for a given a by

$$b \equiv c \pmod{a}.$$

Or b is equivalent to c **modulo** a. A collection of one entry from each set is called a **complete residue system**. We typically select the least positive numbers, those in bold above.

We define arithmetic on congruence classes by arithmetic on the remainders. The remainders wrap around every multiple of the modulus. For example, addition modulo 4 and modulo 5 are defined as follows:

+ (mod 4)	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

+ (mod 5)	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

This works as you expect. Addition is **commutative** and **associative**. There is an **additive identity**, because $b + 0 \equiv 0 \pmod{a}$. Unlike the positive integers, there also is an **additive inverse** for every residue because $b + (a - b) \equiv 0 \pmod{a}$.

Multiplication likewise is **commutative** and **associative**, and there is a **multiplicative identity**, 1. The unusual aspect appears with the **multiplicative inverse**. Some residues have inverses, and some don't:

× (mod 4)	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

× (mod 5)	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

The difference here is that 5 is prime while 4 is composite. Any factor of the modulus will not have a multiplicative inverse.

27.2 Divisibility rules

One common application of modular arithmetic (besides telling time) is in testing whether one integer divides another. We use modular arithmetic and positional notation. Both help us break the larger problem, testing divisibility of a potentially large number, into the smaller problems of breaking apart the number and evaluating expressions in modular arithmetic.

If $a \mid b$ (a divides b), then $b \equiv 0 \pmod{a}$. So we can test for divisibility by expanding b in positional notation and evaluating the operations modulo a.

When the divisor is small, a straight-forward evaluation is simplest. Because $10 \equiv 1 \pmod{3}$, we can test for divisibility by 3 by adding the number's digits modulo 3. For example,

$$\begin{aligned}1234 &\equiv 10^3 + 2\cdot 10^2 + 3\cdot 10 + 4 \pmod{3}\\ &\equiv 1^3 + 2\cdot 1^2 + 3\cdot 1 + 4 \pmod{3}\\ &\equiv 1+2+3+4 \equiv 1+2+0+1 \equiv 1 \pmod{3}.\end{aligned}$$

Hence $3 \nmid 1234$. The same "trick" applies to 9 because $10 \equiv 1 \pmod{9}$.

When the divisor is closer to a power of 10, using a negative element of the congruence class may be useful. For 11, remember that 10 and -1 are in the same congruence class because $10 = 0\cdot 11+10$ and $-1 = -1\cdot 11+10$. So $10 \equiv -1 \pmod{11}$ and we can expand the powers of ten,

$$\begin{aligned}1234 &\equiv 10^3 + 2\cdot 10^2 + 3\cdot 10 + 4 \pmod{11}\\ &\equiv (-1)^3 + 2\cdot(-1)^2 + 3\cdot(-1) + 4 \pmod{11}\\ &\equiv -1+2+-3+4 \pmod{11} \equiv 2 \pmod{11}.\end{aligned}$$

Hence $11 \nmid 1234$. Here, the "trick" form is that you start from the units digit and then alternate subtracting and adding digits.

For more complicated examples, we can factor the divisor. To test if a number is divisible by 72, factor $72 = 2^3\cdot 3^2 = 8\cdot 9$. Then test if the number is divisible by 8 and by 9.

If $a \mid b$ and $c \mid b$, then it **may** be true that $ac \mid b$. This is certainly true of a and b are powers of different primes. The key point is that a and b share no common divisors. Note that $72 = 6\cdot 12$, $6 \mid 24$, and $12 \mid 24$, but obviously $72 \nmid 24$ because $24 < 72$.

27.3 Greatest common divisor

So finding common divisors is useful for testing divisibility. The greatest common divisor of numerator and denominator reduces a fraction into its simplest form.

In general, common divisors help break problems apart.

> Written (a, b) or $\gcd(a, b)$, the greatest common divisor of a and b is the largest integer $d \geq 1$ that divides both a and b.

We'll discuss a total of two methods for finding the greatest common divisor. The first uses the prime factorization, and the second uses the divisibility form in the Euclidean algorithm. Later we'll extend the Euclidean algorithm to provide integer solutions x and y to equations $ax + by = c$.

The prime factorization method factors both a and b. Consider $a = 1400 = 2^3 \cdot 5^2 \cdot 7$ and $b = 1350 = 2 \cdot 3^3 \cdot 5^2$.

Lining up the factorizations and remembering that $x^0 = 1$, we have

$$a = 1400 = 2^3 \cdot 3^{\mathbf{0}} \cdot 5^{\mathbf{2}} \cdot 7^1, \text{ and}$$
$$b = 1350 = 2^{\mathbf{1}} \cdot 3^3 \cdot 5^{\mathbf{2}} \cdot 7^{\mathbf{0}}.$$

Now chose the least exponent for each factor. Then

$$d = 2^1 \cdot 3^0 \cdot 5^2 \cdot 7^0 = 50$$

is the greatest common divisor. For more than two integers, factor all the integers and find the least exponents across the corresponding factors in all of the factorizations.

For an example use, reduce a fraction $a/b = 1350/1400$ to its simplest form. To do so, divide the top and bottom by $d = 50$. Then $a/b = 1350/1400 = 27/28$.

Now we can state the requirement about divisibility given some factors:

> If two relatively prime integers a and b both divide c, then ab divides c.

Some other properties of the gcd:

- Because the gcd is positive, $(a, b) = (|a|, |b|)$.
- $(a, b) = (b, a)$
- If the gcd of two numbers is 1, or $(a, b) = 1$, then a and b are called **relatively prime**.

27.4 Least common multiple

Before the other method for finding the gcd, we consider one related quantity.

> The least common multiple, often written $\operatorname{lcm}(a, b)$, is the least number $L \geq a$ and $L \geq b$ such that $a \mid L$ and $b \mid L$.

There are clear, every day uses. Think of increasing a recipe when you can only buy whole bags of some ingredient. You need to find the least common multiple of the recipe's requirement and the bag's quantity. Or when you need to find the next day two different schedules intersect.

Again, you can work from the prime factorizations

$$a = 1400 = 2^{\mathbf{3}} \cdot 3^0 \cdot 5^{\mathbf{2}} \cdot 7^{\mathbf{1}}, \text{ and}$$
$$b = 1350 = 2^1 \cdot 3^{\mathbf{3}} \cdot 5^{\mathbf{2}} \cdot 7^0.$$

Now the least common multiple is the product of the *larger* exponents,

$$\mathrm{lcm}(a, b) = 2^3 \cdot 3^3 \cdot 5^2 \cdot 7^1 = 37\,800.$$

And for more than two integers, take the maximum across all the exponents of corresponding factors.

Another relation for two integers a and b is that

$$\mathrm{lcm}(a, b) = \frac{ab}{d}.$$

So given $a = 1350$, $b = 1400$, and $d = 50$,

$$\mathrm{lcm}(1350, 1400) = \frac{1350 \cdot 1400}{50} = \frac{1\,890\,000}{50} = 37\,800.$$

This does not hold directly for more than two integers.

27.5 Euclidean GCD algorithm

Another method for computing the gcd of two integers a and b is due to Euclid. This often is called the first algorithm expressed as an abstract sequence of steps.

We start with the division form of b in terms of $a \neq 0$,

$$b = qa + r \text{ with } 0 \leq r < a.$$

Because $(a, b) = (|a|, |b|)$, we can assume both a and b are non-negative. And because $(a, b) = (b, a)$, we can assume $b \geq a$.

Let $d = (a, b)$. Last week we showed that if $d|a$ and $d|b$, then $d|ra + sb$ for any integers r and s. Then because $d|a$ and $d|b$, we have $d|b - qa$ or $d|r$. So we have that $d = (b, a)$ also divides r. Note that any number that divides a and r also divides b, so $d = (a, r)$.

Continuing, we can express a in terms of r as

$$a = q'r + r' \text{ with } 0 \leq r' < r.$$

Now $d|r'$ and $d = (r, r')$. Note that $r' < r < a$, so the problem keeps getting smaller! Eventually, some remainder will be zero. Then the *previous* remainder is the greatest common divisor.

1. Find q_0 and r_0 in $b = q_0 a + r_0$ with $0 \le r_0 < a$.
2. If $r_0 = 0$, then $(a, b) = a$.
3. Let $r_{-1} = a$ to make the loop easier to express.
4. Then for $i = 1, \ldots$
   (a) Find q_i and r_i in $r_{i-2} = q_i r_{i-1} + r_i$ with $0 \le r_i < r_{i-1}$.
   (b) If $r_i = 0$, then $(a, b) = r_{i-1}$ and quit.
   (c) Otherwise continue to the next i.

Consider calculating $(53, 77)$. Following the steps, we have

$$\begin{aligned} 77 &= 1 \cdot 53 + 24, \\ 53 &= 2 \cdot 24 + 5, \\ 24 &= 4 \cdot 5 + 4, \\ 5 &= 1 \cdot 4 + 1, \text{ and} \\ 4 &= 4 \cdot 1 + 0. \end{aligned}$$

And thus $(53, 77) = 1$.

For another example, take $(128, 308)$. Then

$$\begin{aligned} 308 &= 2 \cdot 128 + 52, \\ 128 &= 2 \cdot 52 + 24, \\ 52 &= 2 \cdot 24 + 4, \text{ and} \\ 24 &= 6 \cdot 4 + 0. \end{aligned}$$

So $(128, 308) = 4$.

27.6 Linear Diophantine equations : Likely delayed

Later in the semester, we will examine linear equations $ax + by = c$ over real numbers. But many every-day applications require integer solutions. We can use the Euclidean algorithm to find one integer solution to $ax + by = c$ or prove there are none. Then we can use the computed gcd to walk along the line to all integer solutions.

Say we need to solve $ax + by = c$ for integers a, b, and c to find **integer** solutions x and y. In general, equations over integers are called **Diophantine equations** after Diophantus of Alexandria (approx. 200AD-290AD). He was the first known to study these equations using algebra. The form $ax + b = c$ describes **linear Diophantine equations**.

Let $d = (a, b)$. Then, as before, $d \mid ax + by$ for all integers x and y. So $d \mid c$ for any solutions to exist. If $d \nmid c$, then there are **no integer solutions**. If a and b are relatively prime, then $(a, b) = 1$ and solutions exist for any integer c.

Consider solving $ax + by = d$. Because $d \mid c$, we can multiply solutions to $ax + by = d$ by c/d to obtain solutions of $ax + by = c$. To solve $ax + by = d$ we work backwards after using the Euclidean algorithm to compute $d = (a, b)$.

Say the algorithm required k steps, so $d = r_{k-1}$. Working backward one step,

$$\begin{aligned} d = r_{k-1} &= r_{k-3} - q_{k-1} r_{k-2} \\ &= r_3 - q_{k-1}(r_{k-4} - q_{k-2} r_{k-3}) \\ &= (1 + q_{k-1} q_{k-2}) r_3 - q_{k-1} r_{k-4}. \end{aligned}$$

So $d = r_{k-1} = i \cdot r_{k-3} + j \cdot r_{k-4}$ where i and j are integers. Continuing, the gcd d can be expressed as an integer combination of each pair of remainders.

Returning to the example of $(77, 53)$,

$$\begin{aligned} 1 &= 5 - 1 \cdot 4, \\ &= 5 - 1 \cdot (24 - 5 \cdot 5) = 5 \cdot 5 - 1 \cdot 24 \\ &= 5 \cdot (53 - 2 \cdot 24) - 1 \cdot 24 = 5 \cdot 53 - 11 \cdot 24 \\ &= 5 \cdot 53 - 11 \cdot (77 - 1 \cdot 53) = 16 \cdot 53 - 11 \cdot 77. \end{aligned}$$

To solve $53x + 77y = 22$, we start with $53 \cdot 16 + 77 \cdot (-1) = 1$. Multiplying by 22,

$$53 \cdot (16 \cdot 22) + 77 \cdot (-1 \cdot 22) = 22,$$

and $x = 352$, $y = -22$ is one solution.

But if there is one solution, there are infinitely many! Remember that $d = (a, b)$, so a/d and b/d are integers. Given one solution $x = x_0$ and $y = y_0$, try substituting $x = x_0 + t \cdot (b/d)$ and $y = y_0 - t \cdot (a/d)$ for any integer t. Then

$$\begin{aligned} a(x_0 + t \cdot (b/d)) + b(x_0 - t \cdot (a/d)) &= ax_0 + bx_0 + t \cdot (ab/d)) + -t \cdot (ba/d)) \\ &= ax_0 + bx_0 = c. \end{aligned}$$

Actually, *all* integer solutions to $ax + by = c$ are of the form

$$x = x_0 + t \cdot (b/d), \quad \text{and} \quad y = y_0 - t \cdot (a/d),$$

where t is any integer, $d = (a, b)$, and x_0 and y_0 are a solution pair.

Another example, consider solving $12x + 25y = 331$. First we apply the Euclidian algorithm to compute $(12, 25) = 1$:

$$\begin{aligned} 25 &= 2 \cdot 12 + 1, \text{ and} \\ 12 &= 12 \cdot 1 + 0. \end{aligned}$$

Substituting back,

$$12 \cdot (-2) + 25 \cdot 1 = 1, \text{ and} 12 \cdot (-662) + 25 \cdot 331 \qquad = 331.$$

So we can generate any solution to $12x + 25y = 331$ with the equations

$$x = -662 + 25t \quad \text{and} \quad y = 331 - 12t.$$

Using these, we can find a "smaller" solution. Try making x non-negative with

$$\begin{aligned} -662 + 25t &\geq 0, \\ 25t &\geq 662, \text{ thus} \\ t &> 26. \end{aligned}$$

Substituting $t = 27$,

$$x = 13, \quad \text{and} \quad y = 7.$$

Interestingly enough, this must be the *only* non-negative solution. A larger t will force y negative, and a smaller t forces x negative. But the solution for $t = 26$ is still "small",

$$x = -12, \quad \text{and} \quad y = 19.$$

Chapter 28

Homework for the eighth week: GCD, LCM, $ax + by = c$

28.1 Homework

Notes also available as PDF.

Practice is absolutely critical in this class.

Groups are fine, turn in your own work. Homework is due in or before class on Mondays.

- Exercises 5.3:
  - 62, 66, 70
- Compute the following using **both** the prime factorization method and the Euclidean algorithm:
  - $(720, 241)$
  - $(64, 336)$
  - $(-15, 75)$
- Compute the least common multiples:
  - $\operatorname{lcm}(64, 336)$
  - $\operatorname{lcm}(11, 17)$
  - $\operatorname{lcm}(121, 187)$

  - $\mathrm{lcm}(2025, 648)$

Postponed:

- Find **two** integer solutions to each of the following, or state why no solutions exist:
  - $64x + 336y = 32$
  - $33x - 27y = 11$
  - $31x - 27y = 11$

Note that you *may* email homework. However, I don't use Microsoft$^{\mathrm{TM}}$ products (*e.g.* Word), and software packages are notoriously finicky about translating mathematics.

If you're typing it (which I advise just for practice in whatever tools you use), you likely want to turn in a printout. If you do want to email your submission, please produce a PDF or PostScript document.

Chapter 29

Solutions for eighth week's assignments

Also available as PDF.

29.1 Exercises 5.3

Problem 62 If $(p, q) = p$, then $p \mid q$ and q is a multiple of p.

Problem 66 We want the least common multiple of 6 and 10. The nights off intersect every 30 days. July has 31 days, so by this method their next shared day off is the 31^{st}. On the "trick question" side, though, they may have July 4^{th} off together...

Problem 70 Here we need the greatest common divisor, $(60, 72) = 12$. So the longest common length is 1 foot.

29.2 Computing GCDs

Compute the following using **both** the prime factorization method and the Euclidean algorithm:

- $(720, 241)$
- $(64, 336)$
- $(-15, 75)$

Prime factorizations:

- 241 is prime. So $(720, 241) = 1$.
- $64 = 2^6$, $336 = 2^4 \cdot 3 \cdot 7$. $(64, 336) = 2^4 = 16$.
- $(-15, 75) = (15, 75)$. $15 = 3 \cdot 5$, $75 = 3 \cdot 5^2$, so $(-15, 75) = 15$.

Euclidean algorithm:

- $$\begin{aligned} 720 &= 2 \cdot 241 + 238, \\ 241 &= 1 \cdot 238 + 3, \\ 238 &= 79 \cdot 3 + \mathbf{1} \\ 3 &= 3 \cdot 1 + 0. \end{aligned}$$

  So $(720, 241) = \mathbf{1}$.

- $$\begin{aligned} 336 &= 5 \cdot 64 + \mathbf{16}, \\ 64 &= 4 \cdot 16 + 0. \end{aligned}$$

  So $(336, 64) = 16$.

- $(-15, 75) = (15, 75)$:

  $$75 = 5 \cdot \mathbf{15} + 0.$$

  So $(-15, 75) = 15$.

29.3 Computing LCMs

Compute the least common multiples:

- $\mathrm{lcm}(64, 336)$
- $\mathrm{lcm}(11, 17)$
- $\mathrm{lcm}(121, 187)$
- $\mathrm{lcm}(2025, 648)$

- $\mathrm{lcm}(64, 336) = 64 \cdot 336/(336, 64) = 1344$
- Both are prime, so $\mathrm{lcm}(11, 17) = 11 \cdot 17 = 187$
- $\mathrm{lcm}(121, 187) = \mathrm{lcm}(11^2, 11 \cdot 17) = 11^2 \cdot 17 = 2057$
- $\mathrm{lcm}(2025, 648) = \mathrm{lcm}(3^3 \cdot 5^2, 2^3 \cdot 3^4) = 2^3 \cdot 3^4 \cdot 5^2 = 16200$

Chapter 30

Notes for the ninth week: $ax + by = c$ and fractions

Notes also available as PDF.

30.1 Linear Diophantine equations

In a few weeks, we will examine linear equations $ax + by = c$ over real numbers. But many every-day applications require integer solutions. We can use the Euclidean algorithm to find one integer solution to $ax + by = c$ or prove there are none. Then we can use the computed gcd to walk along the line to all integer solutions.

Some solvable problems:

> A 98 pound box contains 5 pound bags of sugar and 12 pound sacks of oranges. How many of each are in the box?

Or:

> Say you need a digital image in a $4 : 3$ aspect ratio $(x : y)$ that includes a 50 pixel border along each side. What sizes are possible for the inner image?

Consider the latter problem. Rephrasing algebraically,

$$\frac{4}{3} = \frac{x+100}{y+100}, \text{ or}$$
$$3x - 4y = 100.$$

We start by solving

$$3x - 4y = 1$$

and then multiplying the base solutions by 100. This case has one easy solution, $x = -1$ and $y = -1$, with $3 \cdot -1 - 4 \cdot -1 = -3 + 4 = 1$. Another solution is $x = 3$ and $y = 2$.

In fact, there are infintely many solutions to $3x - 4y = 1$ given by

$$\begin{aligned} x &= -1 + 4t, \text{ and} \\ y &= -1 + 3t \end{aligned}$$

for any integer t. You can substitute these expressions into $3x - 4y$ to verify the result. Scaling the right-hand side by 100, solutions to $3x - 4y = 100$ are given by

$$\begin{aligned} x &= -100 + 4t, \text{ and} \\ y &= -100 + 3t. \end{aligned}$$

For $x > 0$ and $y > 0$, we need $t > 33$. So the first positive solutions are given by $t = 34, 35, \ldots$ and are

$$(x, y) \in \{\ldots, (36, 2), (40, 5), (44, 8), (48, 11), (52, 14), (56, 17), (60, 20), \ldots\}.$$

30.1.1 In general...

Say we need to solve $ax + by = c$ for integers a, b, and c to find **integer** solutions x and y. In general, equations over integers are called **Diophantine equations** after Diophantus of Alexandria (approx. 200AD-290AD). He was the first known to study these equations using algebra. The form $ax + b = c$ describes **linear Diophantine equations**.

Let $d = (a, b)$. Then, as before, $d \mid ax + by$ for all integers x and y. So $d \mid c$ for any solutions to exist. If $d \nmid c$, then there are **no integer solutions**. If a and b are relatively prime, then $(a, b) = 1$ and solutions exist for any integer c.

Consider solving $ax + by = d$. Because $d \mid c$, we can multiply solutions to $ax + by = d$ by c/d to obtain solutions of $ax + by = c$. To solve $ax + by = d$ we work backwards after using the Euclidean algorithm to compute $d = (a, b)$.

Say the algorithm required k steps, so $d = r_{k-1}$. Working backward one step,

$$\begin{aligned} d = r_{k-1} &= r_{k-3} - q_{k-1} r_{k-2} \\ &= r_3 - q_{k-1}(r_{k-4} - q_{k-2} r_{k-3}) \\ &= (1 + q_{k-1} q_{k-2}) r_3 - q_{k-1} r_{k-4}. \end{aligned}$$

So $d = r_{k-1} = i \cdot r_{k-3} + j \cdot r_{k-4}$ where i and j are integers. Continuing, the gcd d can be expressed as an integer combination of each pair of remainders.

Returning to the example of $(77, 53)$, we found

$$\begin{aligned}
77 &= 1 \cdot 53 + 24, \\
53 &= 2 \cdot 24 + 5, \\
24 &= 4 \cdot 5 + 4, \\
5 &= 1 \cdot 4 + 1, \text{ and} \\
4 &= 4 \cdot 1 + 0.
\end{aligned}$$

Working from the second to the last,

$$\begin{aligned}
1 &= 5 - 1 \cdot 4, \\
&= 5 - 1 \cdot (24 - 5 \cdot 5) = 5 \cdot 5 - 1 \cdot 24 \\
&= 5 \cdot (53 - 2 \cdot 24) - 1 \cdot 24 = 5 \cdot 53 - 11 \cdot 24 \\
&= 5 \cdot 53 - 11 \cdot (77 - 1 \cdot 53) = 16 \cdot 53 - 11 \cdot 77.
\end{aligned}$$

To solve $53x + 77y = 22$, we start with $53 \cdot 16 + 77 \cdot (-1) = 1$. Multiplying by 22,

$$53 \cdot (16 \cdot 22) + 77 \cdot (-1 \cdot 22) = 22,$$

and $x = 352$, $y = -22$ is one solution.

But if there is one solution, there are infinitely many! Remember that $d = (a, b)$, so a/d and b/d are integers. Given one solution $x = x_0$ and $y = y_0$, try substituting $x = x_0 + t \cdot (b/d)$ and $y = y_0 - t \cdot (a/d)$ for any integer t. Then

$$\begin{aligned}
a(x_0 + t \cdot (b/d)) + b(x_0 - t \cdot (a/d)) &= ax_0 + bx_0 + t \cdot (ab/d)) + -t \cdot (ba/d)) \\
&= ax_0 + bx_0 = c.
\end{aligned}$$

Actually, *all* integer solutions to $ax + by = c$ are of the form

$$x = x_0 + t \cdot (b/d), \quad \text{and} \quad y = y_0 - t \cdot (a/d),$$

where t is any integer, $d = (a, b)$, and x_0 and y_0 are a solution pair.

Another example, consider solving $12x + 25y = 331$. First we apply the Euclidian algorithm to compute $(12, 25) = 1$:

$$\begin{aligned}
25 &= 2 \cdot 12 + 1, \text{ and} \\
12 &= 12 \cdot 1 + 0.
\end{aligned}$$

Substituting back,

$$\begin{aligned}
12 \cdot (-2) + 25 \cdot 1 &= 1, \text{ and} \\
12 \cdot (-662) + 25 \cdot 331 &= 331.
\end{aligned}$$

So we can generate any solution to $12x + 25y = 331$ with the equations

$$x = -662 + 25t \quad \text{and} \quad y = 331 - 12t.$$

Using these, we can find a "smaller" solution. Try making x non-negative with

$$\begin{aligned} -662 + 25t &\geq 0, \\ 25t &\geq 662, \text{ thus} \\ t &> 26. \end{aligned}$$

Substituting $t = 27$,

$$x = 13, \quad \text{and} \quad y = 7.$$

Interestingly enough, this must be the *only* non-negative solution. A larger t will force y negative, and a smaller t forces x negative. But the solution for $t = 26$ is still "small",

$$x = -12, \quad \text{and} \quad y = 19.$$

30.1.2 The other example

Our other posed problem:

> A 98 pound box contains 5 pound bags of sugar and 12 pound sacks of oranges. How many of each are in the box?

So we need to solve $5x + 12y = 98$, and start with $5x + 12y = 1$.

Computing $(12, 5)$,

$$\begin{aligned} 12 &= 2 \cdot 5 + 2, \\ 5 &= 2 \cdot 2 + 1, \text{ and} \\ \\ 2 &= 2 \cdot 1 + 0. \end{aligned}$$

So $(12, 5) = 1$. Because $1 \mid 98$, there are infinitely many integer solutions. We need to find the *non-negative* solutions from those.

For a base solution,

$$\begin{aligned} 1 &= 5 - 2 \cdot 2 \\ &= 5 - 2 \cdot (12 - 2 \cdot 5) \\ &= 5 \cdot (5) + 12 \cdot (-2). \end{aligned}$$

So $x_0 = 5$ and $y_0 = -2$ solve $5x + 12y = 1$. Multiplying by 98,

$$x_0 = 490 \quad \text{and} \quad y_0 = -196$$

solve $5x + 12y = 98$.

To find all solutions,

$$x = 490 + 12t, \text{ and}$$
$$y = -196 - 5t.$$

To find *non-negative* solutions, first consider how to make y positive. Here $t = -40$ makes $y = 4$. Trying x, $x = 10$. So one solution is

$$x_+ = 10 \quad \text{and } y_+ = 5.$$

With $t = -39$, y is negative. And with $t = -41$, x is negative. So this is the **only** possible solution for the actual problem.

30.2 Into real numbers

We've used real numbers without much thought. For the next week and a half, we'll fill in a few details.

- Rational numbers
  - Arithmetic and comparisons
  - Decimal expansion (and other bases)
  - Percentages
- Irrational numbers
  - Show that non-rational real numbers exist
  - Square roots, cube roots, and other radicals
- Computing
  - Floating-point arithmetic (arithmetic with restricted rationals)

This week will cover topics in rational numbers and hence fractions. For some people, this will be old hat. For others, this is a continuing thorn in their sides.

This presentation will be a bit different than the text's more typical structure. I hope that this difference may help some who struggle with rationals and fractions by providing reasons for the rules.

30.2.1 Operator precedence

A quick aside on the order in which operations can be applied. Working with fractions stress operator precedence.

Operations generally don't pass through straight lines, whether horizontal for fractions or vertical for absolute value.

Parentheses force an order. Work from the inner outwards.

The general order of precedence between operations:

1. exponents, then
2. multiplication and division (which really are the same thing), then
3. addition, subtraction, and negation (again, these are the same thing).

Within a class, operations proceed from left to right.

Go through a parenthetical clause and compute every exponent, then every multiplication from left to right, and then every addition from left to right.

When in doubt, use parentheses when you write expressions.

30.3 Rational numbers

Rational numbers are ratios of integers. In a fraction $\frac{n}{d}$, n is the **numerator** and d is the **denominator**. The rational numbers form a set,

$$\mathbb{Q} = \{\frac{a}{b} \,|\, a \in \mathbb{J}, b \in \mathbb{J}, b \neq 0\},$$

where $\mathbb{J}$ is the set of all integers. Let $\mathbb{R}$ be the set of all real numbers, then $\mathbb{J} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$. The integer on top, a, is the **numerator**; the integer on the bottom, b, is the **denominator**.

Note that this is a very formal construction. We just plop one integer atop another and call it a number. Amazing that it works.

Fractions represent ratios and proportions. When you state that 1 in 10 people are attractive to mosquitos[1], that's a rational number. We won't reach probability, where we learn to interpret these ratios correctly, but we will cover basic manipulations of rational numbers.

Two fun points:

- There are only as many rational numbers as there are non-negative integers (and hence integers)! Both sets are infinite, but you can construct a mapping from each non-negative integer to and from a corresponding fraction.
- Between any two real numbers of any sort, there is a rational number. The size of the separation does not matter! There always exists a rational number arbitrarily close to a given real. Consider taking a decimal/calculator expansion and chopping it off once it's close enough.

[1] `http://www.webmd.com/a-to-z-guides/features/are-you-mosquito-magnet`

30.4 Review of rational arithmetic

Rational arithmetic is based on integer arithmetic. The following properties will be inherited by multiplication and addition for rationals q, r, and s:

closure $q + r \in \mathbb{Q}$, $qr \in \mathbb{Q}$

commutative $q + r = r + q$, $qr = rq$,

associative $q + (r + s) = (q + r) + s$, $q(rs) = (qr)s$, and

distributive $q(r + s) = qr + qs$.

One homework question is to take the operation definitions below and verify some of these properties.

The following are somewhat formal definitions to show how to construct rationals along strict rules.

30.4.1 Multiplication and division

We start with multiplication and division. Let $a, b, k, x, y \in \mathbb{J}$, so all the variables are *integers*. We will extend these variables to run over the rational numbers shortly.

The definition of **multiplying fractions**:

$$\frac{a}{b} \cdot \frac{x}{y} = \frac{ax}{by}.$$

As an example

$$\frac{3}{7} \cdot \frac{5}{2} = \frac{15}{14}.$$

The definition of a **relationship between division and fractions**:

$$a/b = a \cdot \frac{1}{b} = \frac{a}{b} \quad \text{so} \quad 8/2 = a \cdot \frac{1}{2} = \frac{8}{2}.$$

We need to be a little careful here. Integer division was defined only when $b \mid a$, so this expression formally only holds when $b \mid a$. We relax this restriction later to allow the variables to run over rational numbers.

An important consequence is that

$$a = \frac{a}{1}$$

for all a.

This leads to very useful technique, **expressing 1 as a fraction**:

$$1 = k/k = \frac{k}{k}$$

for any $k \neq 0$. Remember that in the divisibility form $k = 1 \cdot k + 0$, so $k \mid k$ and $k/k = 1$.

Next we show what the text calls "the fundamental property of rational numbers", which is not terribly fundamental. First we show that 1 is the **multiplicative identity for rationals** by using the fact that 1 is the multiplicative identity for integers,

$$\frac{a}{b} \cdot 1 = \frac{a}{b} \cdot \frac{1}{1} = \frac{a \cdot 1}{b \cdot 1} = \frac{a}{b}.$$

Using this fact, we show that $\frac{a}{b} = \frac{ak}{bk}$ for any $k \neq 0$,

$$\frac{a}{b} = \frac{a}{b} \cdot 1 = \frac{a}{b} \cdot \frac{k}{k} = \frac{ak}{bk}.$$

Now we introduce **proper fractions**. A proper fraction is a rational $\frac{a}{b}$ where the numerator a and denominator b are relatively prime. That is $\gcd(a, b) = 1$ and they share no common factors. Every fraction is equal to some proper fraction. Given $\gcd(a, b) = d$, we can factor out the common divisor,

$$\frac{a}{b} = \frac{a' \cdot d}{b' \cdot d} = \frac{a'}{b'} \cdot \frac{d}{d} = \frac{a'}{b'}.$$

So for 15 and 35, $(15, 35) = 5$ and

$$\frac{15}{35} = \frac{3 \cdot 5}{7 \cdot 5} = \frac{3}{7}.$$

A fraction that is not proper is **improper**. An improper fraction is a redundant representation, and keeping fractions improper sometimes helps speed operations.

Every rational number with a non-zero numerator has a **multiplicative inverse**. This uses only the expression for 1 and the relationship between integer division and fractions. Using that $a/a = 1$ and commutativity of integer multiplication,

$$1 = (ab)/(ab) = \frac{ab}{ab} = \frac{ab}{ba} = \frac{a}{b} \cdot \frac{b}{a}.$$

So if $a \neq 0$, then $\frac{b}{a}$ is the multiplicative inverse of $\frac{a}{b}$. As an example,

$$\frac{3}{5} \cdot \frac{5}{3} = \frac{15}{15} = 1.$$

With a multiplicative inverse, we can define **division of rationals** analogously to the fractional form of division of integers,

$$\frac{a}{b} \Big/ \frac{x}{y} = \frac{a}{b} \cdot \frac{y}{x} = \frac{ay}{bx}.$$

For example,

$$\frac{3}{5} \Big/ \frac{5}{7} = \frac{3}{5} \cdot \frac{7}{5} = \frac{21}{25}.$$

30.4.2 Addition and subtraction

When adding rational numbers, you must ensure both ratios have the same denominators. This is the same as ensuring measurements are all in the same units; both numerators need measured by the same denominator.

The definition for **adding fractions**:

$$\frac{a}{b} + \frac{x}{y} = \frac{ay}{by} + \frac{bx}{by} = \frac{ay + bx}{by}.$$

Later we will use the least common multiple of b and y to work with a smaller initial denominator. So

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}.$$

Rational numbers have additive identities:

$$\frac{a}{b} + \frac{0}{b} = \frac{a+0}{b} = \frac{a}{b}.$$

We prefer there to be only one additive identity. We can use the "fundamental property" above to prove that all additive identities are equal to $\frac{0}{1}$:

$$\frac{0}{b} = \frac{0 \cdot b}{1 \cdot b} = \frac{0}{1} \cdot \frac{b}{b} = \frac{0}{1} \cdot 1 = \frac{0}{1}.$$

Given that $1 \mid 0$, $\frac{0}{1} = 0/1 = 0$. So zero is the **additive identity** for rationals as well as integers.

Like integers, rationals have **additive inverses**:

$$\frac{a}{b} + \frac{-a}{b} = \frac{a + -a}{b} = \frac{0}{b} = 0.$$

Given the additive inverse exists, we can define the **negation** of a rational as

$$-\frac{a}{b} = \frac{-a}{b},$$

and then we define **subtraction** in terms of addition as

$$\frac{a}{b} - \frac{x}{y} = \frac{a}{b} + \frac{-x}{y} = \frac{ay - bx}{xy}.$$

So

$$\frac{1}{2} - \frac{1}{3} = \frac{3}{6} + \frac{-2}{6} = \frac{1}{6}.$$

30.4.3 Comparing fractions

We start with some high-level definitions and find the common product rule for comparing fractions.

First, a quick review of integer ordering. We say an integer is **negative** if it has a negative sign, *e.g.* -1. An integer is **positive** if it is neither zero nor negative, or equivalently if the integer is also a counting number. We start an ordering of the integers by saying that a positive $i > 0$, a negative $i < 0$, and $0 = 0$.

Then we can compare two integers i and j by their difference. There are three cases:

- If $i - j > 0$, then $i > j$.
- If $i - j < 0$, then $i < j$.
- Finally, if $i - j = 0$, then $i = j$.

This phrasing may help with the common confusion regarding comparisons and multiplication by negative numbers.

Consider two integers $3 < 5$. That $3 < 5$ implies $3 - 5 < 0$ (and we know it is -2). Now multiply both sides here by -1. If $3 - 5 < 0$, that implies $3 - 5$ is **negative**, and in turn $-1 \cdot (3 - 5) = 5 - 3 = 2$ is **positive**. *Thus multiplying both sides by -1 (and hence any negative number) requires reversing the comparison.* Here $-1 \cdot (3-5) = 5-3 > 0$. But by the distributive property, $-1 \cdot (3-5) = (-3)-(-5)$ as well, so $(-3) - (-5) > 0$ and $-3 > -5$.

Returning to rationals, the integers are a subset, so an order on the rationals should respect the same ordering on the integer subset.

A positive fraction is equal to some fraction where both the numerator and denominator are positive integers. So both the numerator and denominator must have the *same sign*,

$$\frac{3}{5}, \quad \text{or} \quad \frac{-3}{-5} = \frac{3}{5} \cdot \frac{-1}{-1} = \frac{3}{5}.$$

A negative fraction is equal to some fraction where the numerator is negative and the denominator is positive. Here the signs must be *opposite*,

$$\frac{-3}{5}, \quad \text{or} \quad \frac{3}{-5} = \frac{-3}{5} \cdot \frac{-1}{-1} = \frac{-3}{5}.$$

As we saw with the additive identity, a zero fraction is equal to the integer zero and has a zero numerator. The sign of zero does not matter in rational arithmetic (although it may in a computer's floating-point arithmetic).

Given two rationals r and q, r is strictly less than q, $r < q$, if $q - r$ is positive. Thus

$$\frac{q_n}{q_d} - \frac{r_n}{r_d} = \frac{q_n r_d - q_d r_n}{q_d r_d} > 0.$$

We can always move negative signs into the numerator, so we assume that q_d and r_d are positive. **But remember to convert the fraction into having a positive denominator!** Then the above relation

$$q_n r_d - q_d r_n > 0 \quad \text{or, equivalently,} \quad q_n r_d > q_d r_n.$$

So

$$\frac{q_n}{q_d} > \frac{r_n}{r_d} \quad \text{when} \quad q_n r_d > q_d r_n.$$

By symmetry, then we can compare two rational numbers by comparing appropriate products.

- If $q_n r_d > q_d r_n$, then $\frac{q_n}{q_d} > \frac{r_n}{r_d}$.
- If $q_n r_d < q_d r_n$, then $\frac{q_n}{q_d} < \frac{r_n}{r_d}$.
- If $q_n r_d = q_d r_n$, then $\frac{q_n}{q_d} = \frac{r_n}{r_d}$.

Consider comparing $\frac{1}{2}$ and $\frac{1}{3}$,

$$1 \cdot 3 > 1 \cdot 2 \Rightarrow \frac{1}{2} > \frac{1}{3}.$$

And for the negations $\frac{-1}{2}$ and $\frac{-1}{3}$,

$$-1 \cdot 3 < -1 \cdot 2 \Rightarrow \frac{-1}{2} < \frac{-1}{3}.$$

As an example of why you need to force the denominator to be positive, consider $\frac{1}{2}$ and $\frac{-1}{-3}$.

$$1 \cdot -3 < -1 \cdot 2 \not\Rightarrow \frac{1}{2} < \frac{1}{3}.$$

This is because we are in essence multiplying both sides by the product of their denominators. That product is negative, so we would have to flip the sign. It's just as easy to remember to make the denominator positive.

30.5 Complex fractions

So far, the numerator and denominator have been integers. We can loosen the definition slightly and allow **complex fractions** where the numerator and denominator are rational numbers. We extend the division definition to map complex fractions into fractions with integral numerators and denominators,

$$\frac{\frac{a}{b}}{\frac{x}{y}} = \frac{a}{b} / \frac{x}{y} = \frac{a}{b} \cdot \frac{y}{x} = \frac{ay}{bx}.$$

We could use this definition to show that all the arithmetic operations work as expected on complex fractions.

Working with complex fractions sometimes allows adding fractions without using a massive denominator.

Let L be the least common multiple of b and y. Then $b \mid L$ and $y \mid L$, so L/b and L/y are integers. We can manipulate the addition definition slightly by introducing L,

$$\begin{aligned}
\frac{a}{b} + \frac{x}{y} &= \frac{\frac{a}{b}}{1} + \frac{\frac{x}{y}}{1} = \frac{a \cdot \frac{1}{b}}{1} + \frac{x \cdot \frac{1}{y}}{1} \\
&= \frac{L}{L} \cdot \left(\frac{a \cdot \frac{1}{b}}{1} + \frac{x \cdot \frac{1}{y}}{1} \right) \\
&= \frac{a \cdot \frac{L}{b}}{L} + \frac{x \cdot \frac{L}{y}}{L} \\
&= \frac{a(L/b)}{L} + \frac{x(L/y)}{L} \\
&= \frac{a(L/b) + x(L/y)}{L}.
\end{aligned}$$

With $75 = \operatorname{lcm}(15, 25)$,

$$\frac{7}{15} + \frac{8}{25} = \frac{7 \cdot 3 + 8 \cdot 5}{75} = \frac{61}{75}.$$

Quite often there is less work in reducing the result into proper form if you use the least common multiple as the denominator.

Chapter 31

Homework for the ninth week: $ax + by = c$ and fractions

31.1 Homework

Notes also available as PDF.

Practice is absolutely critical in this class.

Groups are fine, turn in your own work. Homework is due in or before class on Mondays.

- Find **two** integer solutions to each of the following, or state why no solutions exist:
  - $64x + 336y = 32$
  - $33x - 27y = 11$
  - $31x - 27y = 11$
- Problem set 6.3:
  - 5, 6, 9, 10, 13, 14
  - even numbers from 20 up to and including 36
  - 39, 40
  - 57, 58 (alas, I don't think I'll have time to discuss why continued fractions are very interesting and useful)

Note that you *may* email homework. However, I don't use Microsoft$^{\mathrm{TM}}$ products (*e.g.* Word), and software packages are notoriously finicky about translating mathematics.

If you're typing it (which I advise just for practice in whatever tools you use), you likely want to turn in a printout. If you do want to email your submission, please produce a PDF or PostScript document.

Chapter 32

Solutions for ninth week's assignments

Also available as PDF.

32.1 Linear Diophantine equations

> Find **two** integer solutions to each of the following, or state why no solutions exist:
>
> - $64x + 336y = 32$
> - $33x - 27y = 11$
> - $31x - 27y = 11$

- From a previous problem, we have that $336 = 64 \cdot 5 + 16$. Thus $336 \cdot 1 + 64 \cdot -5 = 16$ and $336 \cdot 2 + 64 \cdot -10 = 32$. So one solution is $\mathbf{x_0 = -10}$ and $\mathbf{y_0 = 2}$. The general solution is $x = x_0 + t \cdot 336/(336, 64) = -10 + 21t$ and $y = y_0 - t \cdot 64/(336, 64) = 2 - 4t$ for any integer t. Another solution then is $\mathbf{x(1) = -10 + 1 \cdot 21 = 11}$ and $\mathbf{y(1) = 2 - 4 \cdot 1 = -2}$.
- Here, $(33, 27) = (3 \cdot 11, 3^3) = 3$. Now $3 \nmid 11$, so there are **no solutions**.
- Now 31 is prime, so $(31, 27) = 1 \mid 11$ and there are solutions. Running through the Euclidean algorithm we see that

$$\begin{aligned} 31 &= 27 \cdot 1 + 4, \\ 27 &= 4 \cdot 6 + 3, \text{and} \\ 4 &= 3 \cdot 1 + 1. \end{aligned}$$

Starting from the bottom and substituting for the previous remainder,

$$\begin{aligned} 4 + 3 \cdot (-1) &= 1, \\ 4 + (27 + 4 \cdot (-6)) \cdot -1 = 27 \cdot (-1) + 4 \cdot 7 &= 1, \\ 27 \cdot (-1) + (31 + 27 \cdot (-1)) \cdot 7 = 31 \cdot 7 + 27 \cdot (-8) &= 1. \end{aligned}$$

We find that $31 \cdot 7 + 27 \cdot (-8) = 1$, so $31x - 27y = 11$ has an initial solution of $\mathbf{x_0 = 7 \cdot 11 = 77}$ and $\mathbf{y_0 = -1 \cdot -8 \cdot 11 = 88}$.

The general solutions have the form

$$\begin{aligned} x &= x_0 + t\frac{-27}{(31,27)} = 77 - 27t, \text{ and} \\ y &= y_0 - t\frac{31}{(31,27)} = 88 - 31t, \end{aligned}$$

Another solution is given by $\mathbf{x(1) = 77 - 27 \cdot 1 = 50}$ and $\mathbf{y(1) = 88 - 31 \cdot 1 = 57}$.

32.2 Exercises 6.3

Problem 5 $\frac{16}{48} = \frac{16\cdot 1}{16\cdot 3} = \frac{1}{3}$

Problem 6 $\frac{21}{28} = \frac{7\cdot 3}{7\cdot 4} = \frac{3}{4}$

Problem 9 $\frac{3}{8} = \frac{5\cdot 3}{5\cdot 8} = \frac{15}{40}$, $\frac{3}{8} = \frac{-1\cdot 3}{-1\cdot 8} = \frac{-3}{-8}$, $\frac{3}{8} = \frac{2\cdot 3}{2\cdot 8} = \frac{6}{16}$

Problem 10 $\frac{9}{10} = \frac{-2\cdot 9}{-2\cdot 10} = \frac{-18}{-20}$, $\frac{9}{10} = \frac{2\cdot 9}{2\cdot 10} = \frac{18}{20}$, $\frac{9}{10} = \frac{11\cdot 9}{11\cdot 10} = \frac{99}{110}$

Problem 13
- $\frac{2}{6} = \frac{1}{3}$
- $\frac{1}{4}$
- $\frac{4}{10} = \frac{2}{5}$
- $\frac{3}{9} = \frac{1}{3}$

Problem 14
- $\frac{12}{24} = \frac{1}{2}$
- $\frac{6}{24} = \frac{1}{4}$
- $\frac{12}{16} = \frac{3}{4}$
- $\frac{2}{16} = \frac{1}{8}$

Problem 20 $\frac{8}{9}$

Problem 22 $\frac{41}{90}$

Problem 24 $\frac{14}{60} = \frac{7}{30}$

Problem 26 $\frac{41}{60}$

Problem 28 $\frac{3}{28}$

Problem 30 $\frac{-1}{6}$

Problem 32 $\frac{1}{4}$

Problem 34 $\frac{-3}{10}$

Problem 36 $\frac{-3}{20}$

Problem 39 $\frac{13}{3}$

Problem 40 $\frac{31}{8}$

Problem 57

$$\begin{aligned} 2+\frac{1}{1+\frac{1}{3+\frac{1}{2}}} &= 2+\frac{1}{1+\frac{2}{7}} \\ &= 2+\frac{7}{9} \\ &= \mathbf{\frac{25}{9}} \end{aligned}$$

Problem 58

$$\begin{aligned} 4+\frac{1}{2+\frac{1}{1+\frac{1}{3}}} &= 4+\frac{1}{2+\frac{3}{4}} \\ &= 4+\frac{4}{11} \\ &= \mathbf{\frac{48}{11}} \end{aligned}$$

Chapter 33

Notes for the tenth week: Irrationals and decimals

Notes also available as PDF.

- exponents, roots, and irrationals
- decimals and percentages
- floating-point arithmetic

exponents, roots, and irrationals

- exponents, rules, etc.
- extending to negative exponents: rationals
- extending to rational exponents leads to roots
- roots to/from exponents

33.1 Real numbers

We won't *define* the real numbers. That requires more time than we can allow here. We will simply use the reals, denoted $\mathbb{R}$, as more than the rationals. This was the state of affairs until around 1872 when Richard Dedekind finally discovered a way to construct real numbers formally.

So the reals fit into our system of sets on the very top,

$$\text{natural numbers} \subsetneq \text{whole numbers} \subsetneq \mathbb{J} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}.$$

Look up the term "Dedekind cut" for more on actually defining real numbers.

33.2 Exponents and roots

We will cover:

- definition for positive integer exponents,
- rules,
- zero exponents,
- negative exponents, and
- rational exponents and roots.

33.2.1 Positive exponents

We've already used positive exponents when discussing the digit representation of numbers:

$$\begin{aligned}
10 &= 10 &&= 10^1 \\
100 &= 10 \cdot 10 &&= 10^2 \\
1000 &= 10 \cdot 10 \cdot 10 &&= 10^3 \\
10000 &= 10 \cdot 10 \cdot 10 \cdot 10 &&= 10^4 \\
&\vdots & \vdots && \vdots
\end{aligned}$$

In general, for any number (integer, rational, or real), the number raised to an integer exponent is defined as:

$$\begin{aligned}
a^1 &= a, \\
a^2 &= a \cdot a, \\
a^3 &= a \cdot a \cdot a, \\
&\vdots \\
a^k &= \overbrace{a \cdot a \cdot a \cdot \ldots \cdot a}^{k}.
\end{aligned}$$

For example,

$$\begin{aligned}
2^3 &= 8, \text{ and} \\
\left(\frac{2}{3}\right)^2 &= \frac{4}{9}.
\end{aligned}$$

Negative numbers have signs that bounce around:

$$\begin{aligned}
(-5)^1 &= -5, \\
(-5)^2 &= 25, \\
(-5)^3 &= -125, \text{ and} \\
(-5)^4 &= 625.
\end{aligned}$$

With the symbolic definition, we can show other properties of exponentiation:

$$\begin{aligned}
(ab)^3 &= (ab) \cdot (ab) \cdot (ab) \\
&= (a \cdot a \cdot a) \cdot (b \cdot b \cdot b) \text{ (by commutativity and associativity)} \\
&= a^3 \cdot b^3.
\end{aligned}$$

In general,

$$(ab)^k = a^k b^k.$$

For example,

$$1000 = 10^3 = (2 \cdot 5)^3 = 2^3 \cdot 5^3 = 8 \cdot 125.$$

Or when multiplying numbers raised to powers, we have that exponents add as in

$$\begin{aligned}
a^k \cdot a^m &= \overbrace{a \cdot \ldots \cdot a}^{k} \cdot \overbrace{a \cdot \ldots \cdot a}^{m} \\
&= \overbrace{a \cdot \cdots \cdot a}^{k+m} \\
&= a^{k+m}.
\end{aligned}$$

For example,

$$10^2 \cdot 10^3 = 100 \cdot 1000 = 100000 = 10^5.$$

And numbers raised to powers multiple times multiply exponents as in

$$(a^k)^m = \overbrace{a^k \cdot a^k \cdot a^k \cdot \ldots \cdot a^k}^{m} = a^{km}.$$

For example,

$$100^2 = (10^2)^2 = 10^4 = 10000.$$

33.2.2 Zero exponent

Consider the following relationship between integer exponents and division:

$$\begin{aligned}
a^3 &= a^4/a, \\
a^2 &= a^3/a, \text{ and} \\
a^1 &= a^2/a.
\end{aligned}$$

Reasoning *inductively*, we suspect that

$$a^0 = a^1/a = 1.$$

Using the rule above for adding exponents along with the additive identity property that $k + 0 = k$, we can *deduce* that

$$a^k = a^{k+0} = a^k \cdot a^0.$$

So for any $a \neq 0$,

$$a^0 = 1 \quad \text{when } a \neq 0.$$

Why can't we define this for $a = 0$? $0^k = 0$ for any integer $k > 0$. So $0 = 0^k = 0^k \cdot 0^0$ does not help to define 0^0; we're left with $0 = 0 \cdot 0^0$. Because $0 \cdot x = 0$ for any x, 0^0 can be anything.

Examples:

$$\begin{aligned} 5^0 &= 1 \\ (-73)^0 &= 1 \\ 0^0 &\text{ is undefined} \ldots \end{aligned}$$

33.2.3 Negative exponents

Continuing *inductively* for $a \neq 0$,

$$\begin{aligned} a^0 &= 1, \\ a^{-1} &= a^0/a = \frac{1}{a}, \text{ and} \\ a^{-2} &= a^{-1}/a = \frac{1}{a} \cdot \frac{1}{a} = \frac{1}{a^2}. \end{aligned}$$

Again, we can use the fact that exponents add to derive this *deductively*:

$$1 = a^0 = a^{k+-k} = a^k \cdot a^{-k},$$

and so a^{-k} is the multiplicative inverse of a^k, and we previously showed that to be $\frac{1}{a^k}$. We have shown that

$$a^{-k} = \frac{1}{a^k}$$

for all $a \neq 0$.

For example:

$$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$

is the inverse of

$$2^2 = 4.$$

Also,

$$\begin{aligned}\left(\frac{2}{3}\right)^{-1} &= \frac{1}{\frac{2}{3}}\\ &= \frac{3}{2}\end{aligned}$$

is the multiplicative inverse of

$$\frac{2}{3}.$$

And

$$\begin{aligned}\left(\frac{2}{3}\right)^{-2} &= \frac{1}{(\frac{2}{3})^2}\\ &= \frac{1}{\frac{4}{9}}\\ &= \frac{9}{4}\end{aligned}$$

is the multiplicative inverse of

$$\left(\frac{2}{3}\right)^2 = \frac{2}{3}.$$

33.2.4 Rational exponents and roots

So we've played with division and exponents. Consider now reasoning *inductively* using the multiplication rule for exponents:

$$\begin{aligned}&a^4 = (a^2)^2,\\ &a^2 = (a^1)^2, \text{ and so}\\ &a^1 = (a^{\frac{1}{2}})^2.\end{aligned}$$

We call $a^{\frac{1}{2}}$ the square root of a and write $\sqrt{a}$.

But $\sqrt{a}$ is only defined some of the time. Over integers, there clearly is no integer b such that $b^2 = 2$, so $\sqrt{2}$ is not defined **over the integers** and fractional exponents are **not closed** over integers.

Also, the product of two negative numbers is positive, and the product of two positive numbers is positive, so there is no real number whose square is negative. Hence for real a,

$$\sqrt{a} \text{ is undefined for } a < 0.$$

Remember that $(-b)^2 = (-1)^2 \cdot b^2 = b^2$, so the square root may be either positive or negative!

$$\begin{aligned}
2^2 &= 4, \\
(-2)^2 &= 4, \text{ hence} \\
\sqrt{4} &= \pm 2.
\end{aligned}$$

In most circumstances, $\sqrt{a}$ means the positive root, often called the **principal square root**. When you hit a square-root key or apply a square root in a spreadsheet, you get the principal square root.

Other rationals provide other roots:

$$a^1 = (a^{\frac{1}{3}})^3$$

is the cube root,

$$\sqrt[3]{a} = a^{\frac{1}{3}}.$$

Here, though, $(-a)^3 = (-1)^3 \cdot a^3 = -(a^3)$, and there is no worry about the sign of the cube root.

Using $(a^k)^m = a^{km}$, we also have

$$a^{\frac{2}{3}} = \sqrt[3]{a^2} = (\sqrt[3]{a})^2$$

The exponential operator can be defined on more than just the rationals, but we won't go there. However, remember that I mentioned the rationals are *dense* in the reals. There is a rational number close

33.2.5 Irrational numbers

There are more reals than rationals. This is a very non-obvious statement. To justify it, we will

- prove that $\sqrt{2}$ is not rational, and
- generalize that proof to other roots.

Remember the table to show that there are as many integers as rationals? You cannot construct one for the reals. I might show that someday. It's shockingly simple but still a mind-bender. But for now, a few simple examples suffice to make the point.

Theorem: The number $\sqrt{2}$ is not rational.

Proof. Suppose $\sqrt{2}$ were a rational number. Then

$$\sqrt{2} = \frac{a}{b}$$

for some integers a and b. We will show that any such a and b, two must divide both and so $(a, b) \geq 2$. Previously, we explained that any fraction can be reduced to have $(a, b) = 1$. Proving that $(a, b) \geq 2$ shows that we *cannot* write $\sqrt{2}$ as a fraction.

Now if $\sqrt{2} = \frac{a}{b}$, then $2 = \frac{a^2}{b^2}$ and $2b^2 = a^2$. Because $2 \mid 2b^2$, we also know that $2 \mid a^2$. In turn, $2 \mid a^2$ and 2 being prime imply that $2 \mid a$ and thus $a = 2q$ for some integer q.

With $a = 2q$, $a^2 = 4q^2$. And with $a^2 = 2b^2$, $2b^2 = 4q^2$ or $b^2 = 2q^2$. Now $2 \mid b$ as well as $2 \mid a$, so $(a, b) \geq 2$. □

Theorem: Suppose x and n are positive integers and that $\sqrt[n]{x}$ is rational. Then $\sqrt[n]{x}$ is an integer.

Proof. Because $\sqrt[n]{a}$ is rational and positive, there are positive integers a and b such that

$$\sqrt[n]{x} = \frac{a}{b}.$$

We can assume further that the fraction is in lowest terms, so $(a, b) = 1$. Now we show that $b = 1$.

As in the previous proof, $\sqrt[n]{x} = \frac{a}{b}$ implies that $x \cdot b^n = a^n$.

If $b \geq 1$, there is a prime p that divides b. And as before, $p \mid b$ implies $p \mid a$, contradicting the assumption that $(a, b) = 1$. Thus $b = 1$ and $\sqrt[n]{x}$ is an integer. □

With decimal expansions, we will see that rational numbers have repeating expansions. Irrational numbers have decimal expansions that never repeat. There are some fascinating properties of the expansions

Irrational numbers come in two kinds, **algebraic** and **transcendental**. We won't go into the difference in detail, but numbers like $\sqrt{2}$ are algebraic, and numbers like π and e are transcendental.

33.3 Decimal expansions and percentages

Remember positional notation:

$$1\,234 = 1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0.$$

Given negative exponents, we can expand to the right of 10^0. General English notation uses a **decimal point** to separate the integer portion of the number from the rest.

So with the same notation,

$$\begin{aligned}1\,234.567 &= 1\cdot 10^3 + 2\cdot 10^2 + 3\cdot 10^1 + 4\cdot 10^0\\ &+5\cdot 10^{-1} + 6\cdot 10^{-2} + 7\cdot 10^{-3}.\end{aligned}$$

Operations work in exactly the same digit-by-digit manner as before. When any position goes over 9, a factor of 10 **carries** into the next higher power of 10. If any digit becomes negative, a factor of 10 is **borrowed** frrom the next higher power of 10.

Other languages use a comma to separate the integer from the rest and also use a period to mark off powers of three on the other side, for example

$$1,234.567 = 1.234,567.$$

You may see this if you play with “locales” in various software packages. Obviously, this can lead to massive confusion among travellers. (A price of 1.234 is **not** less than 2 but rather greater than 1000.)

Typical international mathematical and science publications use a period to separate the integer and use a space to break groups of three:

$$1,234.567 = 1\,234.567.$$

33.3.1 Representing rationals with decimals

What is the part to the right of the decimal point? It often is called the **fractional part** of the number, giving away that it is a representation of a fraction.

Here we consider the decimal representation of rational numbers $\frac{1}{a}$ for different integers a. We will see that the expansions fall into two categories:

1. some **terminate** after a few digits, leaving the rest zero; and
2. some **repeat** a trailing section of digits.

For rational numbers, these are the only two possibilities.

We can find the decimal expansions by long division.

Two simple examples that terminate:

$$\begin{array}{r|l} & 0.5 \\ \hline 2 & 1.0 \\ & -1.0 \end{array} \qquad\qquad \begin{array}{r|l} & 0.2 \\ \hline 5 & 1.0 \\ & -1.0 \end{array}$$

Note that $2 \mid 10$ and $5 \mid 10$, so both expansions terminate immediately with $\frac{1}{2} = .5$ and $\frac{1}{5} = .2$.

Actually, all fractions with a denominator consisting of powers of 2 and five have terminating expansions. For example,

$$\begin{aligned} \frac{1}{2^2} &= \frac{1}{4} = 0.25, \\ \frac{1}{5^3} &= \frac{1}{125} = 0.008, \text{ and} \\ \frac{1}{2 \cdot 5^2} &= \frac{1}{50} = 0.02. \end{aligned}$$

What if the denominator a in $\frac{1}{a}$ does not divide 10, or $a \nmid 10$? Then the expansion does not terminate, but it does **repeat**. If the denominator has no factors of 2 or 5, it repeats immediately.

Examples of repeating decimal expansions:

$$\begin{array}{r|l} & 0.33\ldots \\ \hline 3 & 1.000 \\ & -.9 \\ & \;\; .10 \\ & - \;\; 9 \\ & \quad\;\; 10 \end{array} \qquad\qquad \begin{array}{r|l} & 0.1428571\ldots \\ \hline 7 & 1.00000000 \\ & -.7 \\ & \quad 30 \\ & -28 \\ & \qquad 20 \\ & - \quad 14 \\ & \qquad\;\; 60 \\ & - \qquad 56 \\ & \qquad\quad 40 \\ & - \qquad\; 35 \\ & \qquad\qquad 50 \\ & - \qquad\quad 49 \\ & \qquad\qquad\; 10 \\ & - \qquad\qquad\; 7 \\ & \qquad\qquad\quad 3 \end{array}$$

We write these with a bar over the repeating portion, as in

$$\begin{aligned} \frac{1}{3} &= 0.\overline{3}, \text{ and} \\ \frac{1}{7} &= 0.\overline{142857}. \end{aligned}$$

We say that $0.\overline{3}$ has a **period** of 1 and $0.\overline{142857}$ has a period of 6.

We could write $0.2 = 0.2\overline{0}$, but generally we say that this **terminates** once we reach the repeting zeros.

If the denominator a contains factors of 2 or 5, the repeating portion occurs a number of places after the decimal. For example, consider $\frac{1}{6} = \frac{1}{2 \cdot 3}$ and $\frac{1}{45} = \frac{1}{5 \cdot 9}$:

$$\begin{array}{r|l} & 0.166\ldots \\ \hline 6 & 1.0000 \\ & -.6 \\ & \;\;40 \\ & -36 \\ & \;\;\;\;40 \end{array} \qquad\qquad \begin{array}{r|l} & 0.022\ldots \\ \hline 45 & 1.0000 \\ & -.90 \\ & \;\;100 \\ & -\;\;90 \\ & \;\;\;\;10 \end{array}$$

So the decimal representations are

$$\begin{aligned} \frac{1}{6} &= 0.1\overline{6}, \text{ and} \\ \frac{1}{45} &= 0.0\overline{2}. \end{aligned}$$

The hard way to determine the period of a repeating fraction

Note that for all non-negative integer k,

$$\begin{aligned} 10^k &\equiv 0 \pmod 2, \\ 10^k &\equiv 0 \pmod 5, \text{ and} \\ 10^k &\equiv 1 \pmod 3. \end{aligned}$$

These tell us that the expansions have periods of 0, 0, and 1.

For seven,

$$\begin{aligned} 10^0 &\equiv 1 \pmod 7, \\ 10^1 &\equiv 3 \pmod 7, \\ 10^2 &\equiv 2 \pmod 7, \\ 10^3 &\equiv 6 \pmod 7, \\ 10^4 &\equiv 4 \pmod 7, \\ 10^5 &\equiv 5 \pmod 7, \text{ and} \\ 10^6 &\equiv 1 \pmod 7, \end{aligned}$$

so the period is of length 7.

For 45,

$$\begin{aligned} 10^0 &\equiv 1 \pmod{45}, \\ 10^1 &\equiv 10 \pmod{45}, \text{ and} \\ 10^2 &\equiv 10 \pmod{45}. \end{aligned}$$

This is a little more complicated, but the pattern shows that there is one initial digit before hitting a repeating pattern, exactly like the expansion $\frac{1}{45} = 0.0\overline{2}$.

In each case, we are looking for the **order** of 10 modulo the denominator. Finding an integer with a large order modulo another integer is a building block in RSA encryption used in SSL (the `https` prefix in URLs).

33.3.2 The repeating decimal expansion may not be unique!

One common stumbling block for people is that the repeating decimal expansion is not unique.

Let

$$n = 0.\overline{9} = 0.9999\overline{9}.$$

Then multiplying n by 10 shifts the decimal over one but does not alter the pattern, so

$$10n = 9.\overline{9} = 9.9999\overline{9}.$$

Given

$$\begin{aligned} 10n &= 9.\overline{9}, \text{ and} \\ n &= 0.\overline{9}, \end{aligned}$$

we can subtract n from the former.

$$9n = 9.\overline{9} - 0.\overline{9} = 9.$$

With $9n = 9$, we know $n = 1$. Thus $1 = 0.\overline{9}$!

This is a consequence of sums over infinite sequences, a very interesting and useful topic for another course. But this technique is useful for proving that rationals have repeating expansions.

33.3.3 Rationals have terminating or repeating expansions

Theorem: A decimal expansion that repeats (or terminates) represents a rational number.

Proof. Let n be the number represented by a repeating decimal expansion. Without loss of generality, assume that $n > 0$ and that the integer portion is zero. Now let that expansion have d initial digits and then a period of length p. Here we let a terminating decimal be represented by trailing 0 digits with a period of 1.

For example, let $d = 4$ and $p = 5$. Then n looks like

$$n = 0\,.\,d_1 d_2 d_3 d_4 \overline{p_1 p_2 p_3 p_4 p_5}.$$

Then $10^d n$ leaves the repeating portion to the right of the decimal. Following our example $d = 4$ and $p = 5$,

$$10^4 n = d_1 d_2 d_3 d_4\,.\,\overline{p_1 p_2 p_3 p_4 p_5}.$$

Because it repeats, $10^{d+p} n$ has the same pattern to the right of the decimal. In our running example,

$$10^{4+5} n = d_1 d_2 d_3 d_4 p_1 p_2 p_3 p_4 p_5\,.\,\overline{p_1 p_2 p_3 p_4 p_5}.$$

So $10^{d+p} n - 10^d n$ has zeros to the right of the decimal and is an integer k. In our example,

$$k = 10^{4+5} n - 10^4 n = d_1 d_2 d_3 d_4 p_1 p_2 p_3 p_4 p_5 - d_1 d_2 d_3 d_4.$$

We assumed $n > 0$, so the difference above is a positive integer. The fractional parts cancel out.

Now $n = \frac{k}{10^{d+p} - 10^d}$ is one integer over another and thus is rational. □

Theorem: All rational numbers have repeating or terminating decimal expansions.

Proof. This is a very different style of proof, using what we have called the **pidgeonhole principle**. Without loss of generality, assume the rational number of interest is of the form $\frac{1}{d}$ for some positive integer d.

At each step in long division, there are only d possible remainers. If some remainder is 0, the expansion terminates.

If no remainder is 0, then there are only $d-1$ possible remainders that appear. If the expansion is taken to length d, some remainder must appear twice. Because of the long division procedure, equal remainders leave equal sub-problems, and thus the expansion repeats. □

33.3.4 Therefore, irrationals have non-repeating expansions.

So we know that any repeating or terminating decimal expansion represents a rational, and that all rationals have terminating or repeating decimal expansions.

Thus, we have the following:
Corollary: A number is rational if and only if it has a repeating decimal expansion.

So if there is no repeating portion, the number is *irrational*. One example,

$$0.101001000100001\cdots,$$

has an increasing number of zero digits between each one digit. This number is irrational.

It's beyond our scope to prove that π is irrational, but it is. Thus the digits of π do not repeat.

33.3.5 Percentages as rationals and decimals

Percentage comes from *per centile*, or part per 100. So a direct numerical equivalent to 85% is

$$85\% = \frac{85}{100} = .85.$$

We can expand fractions to include decimals in the numerator and denominator. The decimals are just rationals in another form, and we already explored "complex fractions" with rational numerators and denominators.

So we can express decimal percentages,

$$85.75\% = \frac{85.75}{100} = .8575.$$

Everything else "just works". To convert a fraction into a percentage, there are two routes. One is to convert the denominator into 100:

$$\frac{1}{2} = \frac{50}{100} = 50\%.$$

Another is to produce the decimal expansion and then multiply that by 100:

$$\frac{1}{7} = 0.\overline{142857} = 14.2857\overline{142857}\%.$$

Converting a percentage into a proper fraction required dropping the percentage into the numerator and then manipulating it appropriately:

$$85.75\% = \frac{85.75}{100} = \frac{\frac{8575}{100}}{100} = \frac{8575}{10000} = \frac{343}{400}.$$

33.4 Fixed and floating-point arithmetic

So far we have considered *infinite* expansions, ones that are not limited to a set number of digits. Computers (and calculators) cannot store infinite expansions that do not repeat, and those that do require more overhead than they are worth.

Instead, computers **round** infinite results to have at most a fixed number of **significant digits**. Operations on these limited representations incur some **round-off error**, leading to a tension between computing speed and the precision of computed results. One important fact to bear in mind is that **precision does not imply accuracy**. The following is a very precise but completely in-accurate statement:

> The moon is made of Camembert cheese.

First we'll cover different rounding rules from the perspective of **fixed-point arithmetic**, or arithmetic using a set number of digits to the right of the decimal plce. Then we'll explain **floating-point arithmetic** where the decimal point "floats" through a fixed number of significant digits.

We will not cover the errors in floating-point operations, but we will cover the errors that come from typical binary representation of decimal data.

The points you need to take away from this are the following:

- Using a limited number of digits (or bits) to represent real numbers leads to some inherent, representationall error.
- Representing every-day decimal quantities in binary also incurs some representational error.

Despite the doom-like points above, floating-point arithmetic often provides results that are accurate enough. We won't be able to cover why this is, but the high-level reasons include:

- using far more digits of precision than initially appear necessary, and
- carrying intermediate results to even higher precision.

33.4.1 Rounding rules

Generally, computer arithmetic can be modelled as computing the **exact** result and then rounding that exact result into an economical representation.

truncation or rounding to zero With this rounding method, digits beyond the stored digits are simply dropped.

rounding half-way away from zero This is the text's method of "round half up". A number is rounded to the nearest representable number. In the half-way case, where the digits beyond the number of digits stored are $5000\cdots$, the number is rounded upwards.

rounding half-way to even This is the **preferred** method for rounding in general. A number is rounded to the nearest representable number. In the half-way case, where the digits beyond the number of digits stored are $5000\cdots$, the number is rounded so the final stored digit is **even**.

There are more rounding methods, but these suffice for our discussion. Rounding rules are hugely important in banking and finance, and there are quite a few versions required by different regulations and laws.

Examples of each rounding method above, rounding to two places after the decimal point:

initial number	truncate	round half up	round to nearest even
$\frac{1}{3} = 0.\overline{3}$	0.33	0.33	0.33
$\frac{1}{7} = 0.\overline{142857}$	0.14	0.14	0.14
0.444	0.44	0.44	0.44
0.445	0.44	0.45	0.44
0.4451	0.44	0.45	0.45
0.446	0.44	0.45	0.45
0.455	0.44	0.46	0.46

Rounding error is the absolute difference between the exact number and the rounded, stored representation. In the table above, the rounding error in representing $\frac{1}{3}$ is $|\frac{1}{3} - 0.33| = |\frac{1}{3} - \frac{33}{100}| = |\frac{100}{300} - \frac{99}{300}| = \frac{1}{300} = 0.00\overline{3}$. Note that here the rounding error is 1% of the exact result. That error is large because we use only two digits.

Note that you cannot round in stages. Consider round-to-nearest-even applied to 0.99455 and rounding to two places after the point:

Incorrect	Correct
0.99455	0.99455
0.9946	
0.995	
1.00	0.99

33.4.2 Floating-point representation

Consider repeatedly dividing by 10 in fixed-point arithmetic that carries two digits beyond the decimal:

$$\begin{aligned} 1 \div 10 &= 0.10, \\ 0.1 \div 10 &= 0.01, \\ 0.01 \div 10 &= 0.00. \end{aligned}$$

So $((1 \div 10) \div 10) \div 10$ evaluates to 0! This phenomenon is called **underflow**, where a number grows too small to be represented. A similar phenomenon, **overflow**, occurs when a number becomes too large to be represented. Computer arithmetics differ on how they handle over- and underflow, but generally overflow produces an ∞ symbol and underflow produces 0.

Floating-point arithmetic compensates for this by carrying a fixed number of **significant** digits rather than a fixed number of fractional digits. The position of the decimal place is carried in an **explicit, integer exponent**. This allows floating-point numbers to store a wider range and actually makes analysis of the round-off error easier.

In floating-point arithmetic,

$$\begin{aligned}
1 \div 10 &= 1.000 \cdot 10^{0}, \\
0.1 \div 10 &= 1.000 \cdot 10^{-1}, \\
0.01 \div 10 & 1.000 \cdot 10^{-2}, \\
&\vdots
\end{aligned}$$

This continues until we run out of representable range for the integer exponents. We leave the details of floating-point underflow for another day (if you're unlucky).

33.4.3 Binary fractional parts

Just as integers can be converted to other bases, fractional parts can be converted as well.

Each position to the right of the point (no longer the *decimal* point) corresponds to a power of the base. For binary, the typical computer representation,

$$\begin{aligned}
\frac{1}{2} &= 2^{-1} = 0.1_2 = 0.5, \\
\frac{1}{4} &= 2^{-2} = 0.01_2 = 0.25, \\
\frac{1}{8} &= 2^{-3} = 0.001_2 = 0.125.
\end{aligned}$$

So a binary fractional part can be expanded with powers of two:

$$0.1101_2 = \frac{1}{2^1} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} = 0.8125.$$

To find a binary expansion, we need to carry out long division in base 2. I won't ask you to do that.

The important part to recognize is that finite *decimal* expansions may have infinite, repeating *binary* expansions! Remember that in decimal, $2 \mid 10$ and $5 \mid 10$, so negative powers of 2 and 5 have terminating decimal expansions. In binary, only $2 \mid 2$, so only powers of 2 have terminating binary expansions.

Numbers you expect to be exact are not. Consider 0.1. Its binary expansion is

$$0.1 = 0.\overline{00011}_2.$$

A five-bit fixed-point representation would use

$$0.1 \approx 0.00011_2.$$

The error in representing this with a five-digit fixed-point representation is 0.00625, or over 6%.

In a five-bit floating-point representation,

$$0.1 \approx 1.1001_2 \cdot 2^{-4}.$$

The error here is less than 0.0024, or less than 0.24%. You can see what floating-point gains here.

Ultimately, though, in a limited binary fractional representation, adding ten dimes does not equal one dollar! This is why often programs slanted towards finance (*e.g.* spreadsheets) use a form of decimal arithmetic. On current common hardware, decimal arithmetic is implemented in software rather than hardware and is orders of magnitude slower than binary arithmetic.

Chapter 34

Homework for the tenth week: Irrationals and decimals

34.1 Homework

Notes also available as PDF.

Practice is absolutely critical in this class.

Groups are fine, turn in your own work. Homework is due in or before class on Mondays.

- Problem set 6.4:
  - 15, 16
  - 25-28 (Note: If you write these as $a^{\frac{1}{2}}$, you can use the rule $(ab)^k = a^k b^k$ and the factorization of each number to simplify the expressions. That summarizes the text's examples.)
  - 41-44 (Again, think of these as fractional exponents. Use the factorizations of each number under the square root, and then use the distributive property to pull out common factors.)
  - 49
  - 81
- Problem set 6.3:
  - 75, 76, 79, 80 (calculators are fine, but correctly denote what repeats)

  - 86, 87, 88
  - 95, 96
- Problem set 6.5:
  - 1-5
- On rounding and floating point arithmetic:
  - Round each of the following to the nearest tenth (one place after the decimal) using **round to nearest even**, **round to zero (truncation)**, and **round half-up**:
    * 86.548
    * 86.554
    * 86.55
  - Compute the following quantities with a computer or a calculator. **Write what type of computer/calculator you used and the software package if it's a computer.** Compute it as shown. Do not simplify the expression before computing it, and do not re-enter the intermediate results into the calculator or computer program. Also compute the expressions that do not include 10^{16} by hand exactly. There should be a difference between the exact result and the displayed result in some of these cases. Remember to work from the innermost parentheses outward.
    * $(\overbrace{0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1}^{\text{10 times}})-1$
    * $((\overbrace{0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1}^{\text{10 times}})-1)\times 10^{16}$, where 10^{16} often is entered as 1e16. If the result overflows (signals an error) on various calculators, replace 10^{16} by 10^{8} in this and later portions.
    * $(((2 \div 3) - 1) \times 3) + 1$
    * $((((2 \div 3) - 1) \times 3) + 1) \times 10^{16}$
    * $(((6 \div 7) - 1) \times 7) + 1$
    * $((((6 \div 7) - 1) \times 7) + 1) \times 10^{16}$

    The object of this first part is to demonstrate round-off error. The first to problems, adding 0.1 repeatedly, may see no error if the device calculates in decimal. The latter four parts should see some error regardless of the base used.
  - Now copy down the number displayed by the first calculation in each of the following. Re-enter it as x in the second calculation.

* $1 \div 3$, then $1 \div 3 - x$ where x is the number displayed.
* If you have a calculator or program with π, π, then $\pi - x$ where x is the number displayed.

The object here is to see that the number displayed often is not the number the computer or calculator has stored.

Note that you *may* email homework. However, I don't use Microsoft™ products (*e.g.* Word), and software packages are notoriously finicky about translating mathematics.

If you're typing it (which I advise just for practice in whatever tools you use), you likely want to turn in a printout. If you do want to email your submission, please produce a PDF or PostScript document.

Chapter 35

Solutions for tenth week's assignments

Also available as PDF.

35.1 Exercises 6.4

Problem 15 The sum is $0.\overline{8}$, which is rational. So the sum of two irrationals may be rational. That should not be surprising; $2-\sqrt{2}$ and $\sqrt{2}$ are irrational, but their sum is the rational 2.

Problem 16 The sum is $0.262662666\ldots$, which is not repeating or terminating and thus is irrational. So the sum of two irrationals, like $\sqrt{2}$ and $\sqrt{2}$, can be irrational, like $2\sqrt{2}$.

Problem 25 $\sqrt{50} = \sqrt{2\cdot 5^2} = (2\cdot 5^2)^{\frac{1}{2}} = 5\sqrt{2} \approx 7.07$.

Problem 26 $\sqrt{32} = \sqrt{2^5} = 2^2\sqrt{2} \approx 5.66$

Problem 27 $\sqrt{75} = \sqrt{3\cdot 5^2} = 5\sqrt{3} \approx 8.66$

Problem 28 $\sqrt{150} = \sqrt{2\cdot 3\cdot 5^2} = 5\sqrt{6} \approx 12.25$.

Problem 41 $3\sqrt{18}+\sqrt{2} = 3\cdot(2\cdot 3^2)^{\frac{1}{2}} + 2^{\frac{1}{2}} = 9\cdot 2^{\frac{1}{2}} + 2^{\frac{1}{2}} = 10\cdot 2^{\frac{1}{2}} = 10\sqrt{2}$

Problem 42 $2\sqrt{48}-\sqrt{3} = 2\cdot(2^4\cdot 3)^{\frac{1}{2}} - 3^{\frac{1}{2}} = 2^3\cdot 3^{\frac{1}{2}} - 3^{\frac{1}{2}} = 7\sqrt{3}$

Problem 43 $-\sqrt{12}+\sqrt{75} = -(2^2\cdot 3)^{\frac{1}{2}} + (3\cdot 5^2)^{\frac{1}{2}} = -2\cdot 3^{\frac{1}{2}} + 5\cdot 3^{\frac{1}{2}} = (5-2)\sqrt{3} = 3\sqrt{3}$

Problem 44 $2\sqrt{27}-\sqrt{300} = 2\cdot(3^3)^{\frac{1}{2}} - (3\cdot 10^2)^{\frac{1}{2}} = 6\cdot 3^{\frac{1}{2}} - 10\cdot 3^{\frac{1}{2}} = -4\sqrt{3}$

Problem 49 $P = 2\pi\sqrt{\frac{5.1}{32}} \approx 2.5$. Note that 32 is an approximation to gravitational acceleration.

Problem 81 Approximating:

	1.1^{10}	1.01^{100}	1.001^{1000}	1.0001^{10000}	1.00001^{100000}
Approximating:	2.5937	2.7048	2.7169	2.7181	2.7183
	$0.95418e$	$0.99505e$	$0.99950e$	$0.99095e$	$1.00000e$

The constant $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$, and the experimental results above bear this out. The first few digits converge very quickly.

35.2 Exercises 6.3

Problem 75 0.75

Problem 76 0.875

Problem 79 $0.\overline{27}$

Problem 80 $0.\overline{81}$

Problem 86 $0.105 = \frac{105}{1000} = \frac{21}{200}$

Problem 87 $0.934 = \frac{934}{1000} = \frac{467}{500}$

Problem 88 $0.7984 = \frac{7984}{10000} = \frac{499}{625}$

Problem 95

- $\frac{1}{3} = 0.\overline{3}$
- $\frac{2}{3} = 0.\overline{6}$
- $\frac{1}{3} + \frac{2}{3} = 0.\overline{3} + 0.\overline{6} = 0.\overline{9}$
- As covered in class, repeating nines are equal to a one one digit over, so $0.\overline{9} = 1$.

Problem 96 Here, $3 \cdot 0.\overline{3} = 0.\overline{9} = 1$.

35.3 Exercises 6.5

Problem 1 $3.00 \cdot 12 = 36$, **true**

Problem 2 $0.25 = \frac{25}{100} = \frac{1}{4}$, **true**

Problem 3 Rounding 759.367 to the second place beyond the decimal should give either 759.37 or 759.36 depending on the rounding rule. **false**

Problem 4 With round-to-nearest (even or upwards), this is **true**.

Problem 5 $0.50 = \frac{50}{100} = \frac{1}{2}$, **true**

35.4 Rounding and floating-point

35.4.1 Rounding

Round each of the following to the nearest tenth (one place after the decimal) using **round to nearest even**, **round to zero (truncation)**, and **round half-up**:

- 86.548
- 86.554
- 86.55

Number	Round to nearest even	Truncate	Round half-up
86.548	86.5	86.5	86.5
86.554	86.6	86.5	86.6
86.55	86.6	86.5	86.6

35.4.2 Errors in computations

Compute the following quantities with a computer or a calculator. **Write what type of computer/calculator you used and the software package if it's a computer.** Compute it as shown. Do not simplify the expression before computing it, and do not re-enter the intermediate results into the calculator or computer program. Also compute the expressions that do not include 10^{16} by hand exactly. There should be a difference between the exact result and the displayed result in some of these cases. Remember to work from the innermost parentheses outward.

- $(\overbrace{0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1}^{\text{10 times}})-1$
- $((\overbrace{0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1}^{\text{10 times}})-1)\times 10^{16}$, where 10^{16} often is entered as 1e16. If the result overflows (signals an error) on various calculators, replace 10^{16} by 10^8 in this and later portions.
- $(((2 \div 3) - 1) \times 3) + 1$
- $((((2 \div 3) - 1) \times 3) + 1) \times 10^{16}$
- $(((6 \div 7) - 1) \times 7) + 1$
- $((((6 \div 7) - 1) \times 7) + 1) \times 10^{16}$

> The object of this first part is to demonstrate round-off error. The first to problems, adding 0.1 repeatedly, may see no error if the device calculates in decimal. The latter four parts should see some error regardless of the base used.

Using Octave on a 64-bit Intel-based machine with the "short" display format:

- $-1.1102 \cdot 10^{-16}$
- -1.1102
- 0 : Sometimes errors cancel themselves out. Not every computational error is **bad**.
- 0
- $-4.4409 \cdot 10^{-16}$
- -4.4409

35.4.3 Extra digits

> Now copy down the number displayed by the first calculation in each of the following. Re-enter it as x in the second calculation.
>
> - $1 \div 3$, then $1 \div 3 - x$ where x is the number displayed.
> - If you have a calculator or program with π, π, then $\pi - x$ where x is the number displayed.
>
> The object here is to see that the number displayed often is not the number the computer or calculator has stored.

Using Octave on a 64-bit Intel-based machine with the "short" display format:

- `1 / 3` produces `0.33333`. Then `(1/3) - 0.33333` produces `3.3333e-06` or $3.3333 \cdot 10^{-6}$.
- `pi` produces `3.1416`, and `pi - 3.1416` produces `-7.3464e-06` or $-7.3464 \cdot 10^{-6}$.

Chapter 36

Second exam and solutions

Available as PDF.

Part IV

Notes for chapters 7 and 8

Chapter 37

Notes for the twelfth week

(Still in progress)

Notes also available as PDF. ***For now, graphs are only in the PDF version.***

37.1 Covered So Far

- Problem solving techniques
  - Pólya's principles:
    1. *Understand the problem*
    2. Make a plan
    3. Carry out the plan
    4. Look back at what you've done
  - We will be covering algebra and graphs as means for rephrasing and understanding problems.
- Technical vocabulary for mathematics: sets and logic
  - We need a basic vocabulary for describing mathematical entities.
  - Will be using sets to describe equation solutions and functions.
  - Logic underpins everything.
- Number sense and operations from number theory and rational arithmetic
  - Gain some feeling for when numbers and solutions make sense.
  - Is an even number possible? Negative number? *etc.*

37.2 What Will Be Covered

Topic: Algebra and graphs for mathematical modeling.

A **mathematical model** is a mechanism used for predicting responses from data.

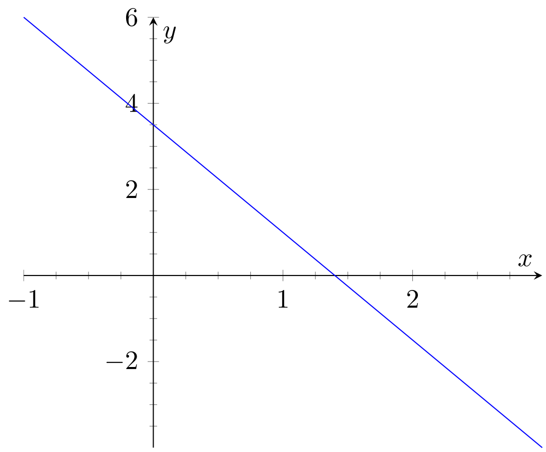
- A climate modefl is a simulation that takes data (mostly satellite sensor readings) and generates concrete predictions.
- A population model often is a formula that takes a limited set of data (*e.g.* initial sizes) and produces a rough estimate of how the populations grow and shrink.

Consider these real life "word problems." The models often can be expressed algebraically; we will be covering a few of these forms.

Sometimes an **algebraic** view, working with symbols, is the most useful, and sometimes a **graphical** view, working with plots, is the most useful. Different people and different problems may require different views for fully understanding them. Regardless, each view can serve as a good check.

The algebraic models and relationships we will cover:

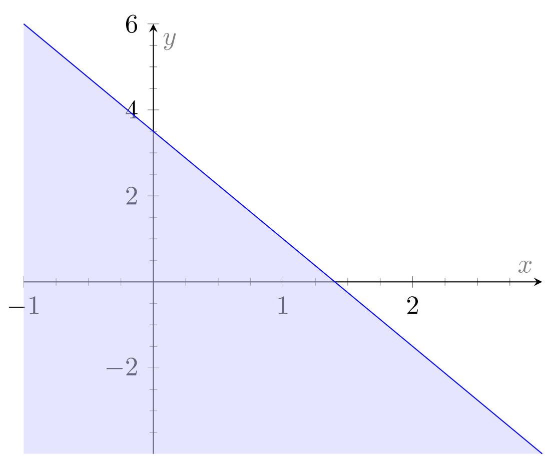
- Linear equations
  - Equations in one variable: Useful for simple modeling and for describing algebraic rules. Examples:
    * $x = 5$
    * $65x + 39 = 364$
  - Equations in multiple variables: Lines, planes, hyperplanes. Examples:
    * $5x + 2y = 7$



- Inequalities: Constraints on values. Examples:

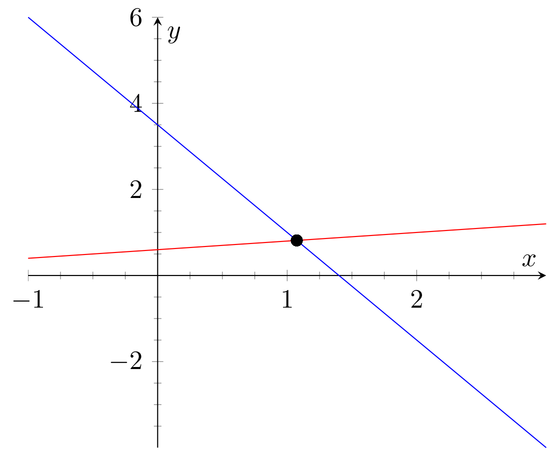
  * $5x \leq 7$

  * $5x + 2y \leq 7$

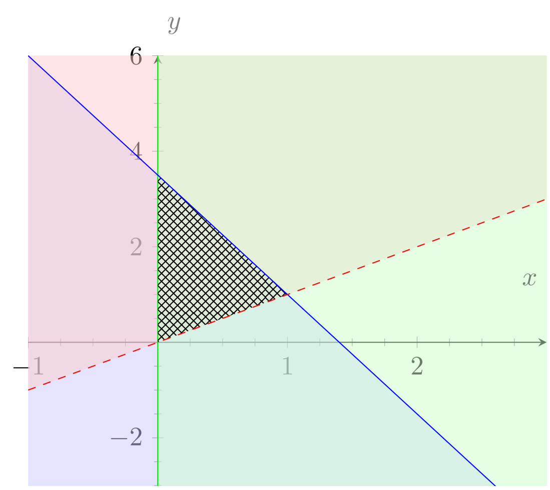


- Systems: Multiple equations or inequalities. Examples:

  * $5x + 2y = 7$, $-x + 5y = 3$

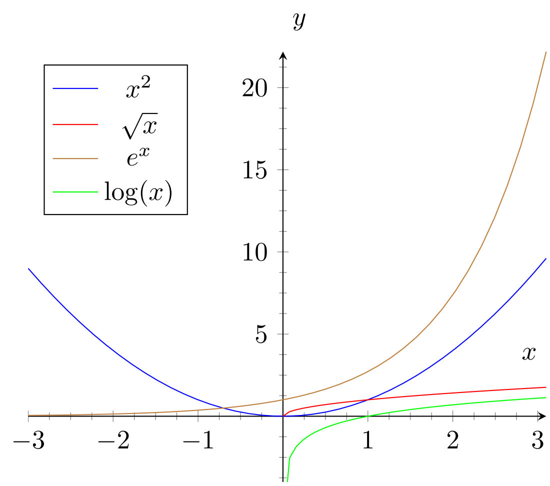


∗ $5x + 2y \leq 7$, $x \geq 0$, $y > x$



- Non-linear equations

  – In one variable: powers, roots, and logarithms



37.3 An Algebraic Example

Consider the following problem:

> Some proposition loses by 4 votes out of 100 votes cast. How many voted *yes* and how many voted *no*?

One of the first questions to answer is if there is enough information to find a solution. The answer here is *yes*, although it may not be obvious from the description above.

For now, we approach this problem **algebraically** using symbols. This is the approach you remember for general word problems. I won't explain everything here; this is an example we can use throughout. Afterwards, I'll describe a **graphical** approach we can use to quickly see if there is any hope of solution.

The first step in an algebraic approach is to assign **variables** to the unknowns. Here, let Y be the number of *yes* votes, and let N be the number of *no* votes. For now, we do not worry about requiring these to be integers or non-negative numbers. One of the techniques in modeling and algebra is knowing what to ignore and when to ignore it. Often, you ignore some properties of the data. Once you have a final result or solution, you can re-apply those properties to see if it makes sense.

Following Pólya's principles, we next *rephrase the problem* by relating the variables with **equations**.

$$\begin{aligned} \text{loses by 4 votes:} \quad & N = Y + 4 \\ \text{total of 100 votes:} \quad & Y + N = 100 \end{aligned}$$

The general algebraic method for solving systems of equations is to **simplify** them into forms that lead to a result. This is a general *plan* for solving algebraic problems, but it needs broken into sub-plans that may not be obvious when you start.

Here, we can reduce the problem over two variables, Y and N, into a problem over one variable, N, by **substituting** the first equation into the second's left-hand side:

$$\begin{aligned}
Y + N &= Y + (Y + 4) && \text{by substitution} \\
&= (Y + Y) + 4 && \text{by the associative property} \\
&= 2Y + 4 && \text{by evaluating } Y + Y.
\end{aligned}$$

Now we substitute $2Y + 4$ for $Y + N$ and transform both sides of $2Y + 4 = 100$:

$$\begin{aligned}
2Y + 4 &= 100, && \text{inital equations} \\
(2Y + 4) - 4 &= 100 - 4, && \text{subtract the same from equal quantities} \\
2Y + (4 - 4) &= 96, && \text{associative property} \\
2Y + 0 &= 96, && \text{additive inverse} \\
2Y &= 96, && \text{additive identity} \\
\frac{1}{2} \cdot 2Y &= \frac{1}{2} \cdot 96, && \text{mult. equal quantities by the same} \\
1 \cdot Y &= 48, && \text{mult. inverse and evaluation} \\
Y &= 48, && \text{mult. identity.}
\end{aligned}$$

Then $Y = 48$ and $N = 100 - 48 = 52$. So the final path by which we solved the problem:

1. Rephrase the problem algebraically.
2. Substitute to eliminate one variable, simplifying the problem.
3. Solve a linear equation in one variable.
4. Then substitute back to obtain the other variable.

One of the primary topics we will cover is when to skip all the intermediate steps. Many of the algebra rules we and the text present are designed to allow skipping from the two equations directly to $2Y + 4 = 100$ and then more quickly to $Y = 48$. The reasons given above are purely general, and algebraic notation provides the means of applying those reasons generally.

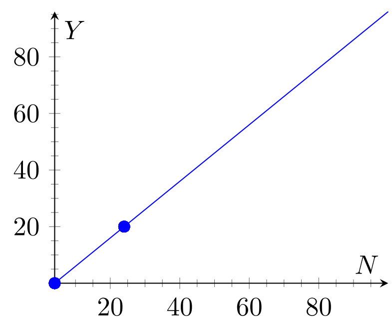
Note that the answer makes sense according to what we know of the problem domain. You cannot have fractional or negative votes. The results are positive integers, so they make sense.

For an example where the number theory we covered helps identify a incorrect data, consider a slight variation where $N = Y + 3$. From the solution above, we know that the difference will satisfy $2Y + 3 = 100$. If Y is an integer, we know that $2 \mid 2Y$ and $2 \mid 100$. But $2 \nmid 3$, so we know this equality cannot have an integer solution. So if our initial method is carried out correctly, *i.e.* we substituted $N = Y + 3$ into $N + Y = 100$ correctly, we know that the problem's initial data *must* be incorrect. This is what I mean by **number sense**.

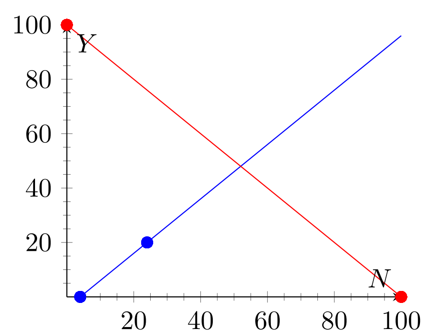
37.4 The Example's Graphical Side

Thinking of the equations above as relationships, we can plot them on an N-Y graph. I will go into lines and linear forms later. For now, it suffices to remember Euclid's axiom that two points define a line.

For $N = Y + 4$, consider the points where $Y = 0$ and $Y = 20$. These give the values $N = 4$ and $N = 24$. Drawing a line between these points



Now consider $N + Y = 100$. Two points suggest themselves immediately, one at $N = 0$ and one at $Y = 0$. Adding a line through these points:



Even just sketching these without being two precise shows us that the solution may make sense. One line slopes up and one slopes down, so they will intersect somewhere. And a quick sketch shows they intersect with both variables taking positive numbers. This style of **graphical reasoning** often helps show if a solution is possible or impossible. A quick sketch does not show that the variables are positive *integers*, but the sketch does justify carrying out the algebraic work.

37.5 Definitions

Now for the painful part. We need a common set of definitions.

- An **algebraic expression** is a phrase containing variables, numbers, operations, and groups (parentheses). Examples:
  - $5x$
  - $763 + 873672 + (-77 + 232)$
  - $\sqrt{52x + \sqrt{11 + \sqrt{22y}}}$
- An **equation** relates two algebraic expressions by equality. In some contexts, these also are **equalities** or **identities**. Examples:
  - $5x = 763 + 873672 + (-77 + 232)$
  - $\sqrt{52x + \sqrt{11 + \sqrt{22y}}} = 77z$
- An **inequality** relates two algebraic expressions by a comparison. Examples:
  - $5x < 763 + 873672 + (-77 + 232)$
  - $\sqrt{52x + \sqrt{11 + \sqrt{22y}}} \geq 77z$
- More generally, equations and inequalities are called **relations**. Relations also include negated equalities and inequalities ($x \neq 4$, $36x + 93 \not> 39$).
- A **variable** is a symbol standing for a number or quantity. Variables can be **known** or **unknown**.
  - When solving $5x + 7 = 83$, the variable x is considered an **unknown**.
  - But sometimes repeating a long expression, *e.g.* $\frac{\sqrt{238x+\sqrt{281y}}}{98z}$, becomes cumbersome, and you replace it with with a **known** variable rather than writing it again and again.
- The **degree of a variable** in an expression is the largest exponent applied to the variable. Examples:
  - x has degree one, or is first degree;
  - x^2 has degree two, or is second degree; *etc.*

  The **degree of an expression** is the largest degree of any variable in that expression. Examples:
  - $5x + 8$ has degree one, or is first degree;
  - $x^2 + 5x + 8$ has degree two, or is second degree;
  - $29x^2 + 38219y + 91z^3$ has degree three because of z^3.

There is an unusual corner-case where the context matters. In an expression like x^2y^3, the degree can be two, three, or five depending on other constraints and considerations. If you know nothing more about the context, then often x^2y^3 is considered of third degree because of the y^3 term. We won't worry about these situations.

- The **solution set** is the set of solutions, or the set of variable values that satisfy the given relation (equation, inequality, *etc.*). Examples:
  - $x = 5$ has a solution set of $\{5\}$ for x.
  - $N = Y + 3$, $N + Y = 100$ has the solution set $\{(52, 48)\}$ for the *pair* (N, Y).
- **Equivalent equations** are equations with the same solution sets. Solving equations algebraically consists of transforming equations into simpler, equivalent equations. For example $2Y + 4 = 100$ is equivalent to $2Y = 96$ and $Y = 48$.

  Again, a bit of context can make a difference. For example, $x = 5$ and $y = 5$ have the same solution set, but there are contexts where they are not truely equivalent. We could define equivalence in a way to handle this (*c.f.* β-reduction (beta reduction) in programming language theory), but that's beyond our scope.
- **Equivalent inequalities** and relations are defined similarly.

37.6 Algebraic Rules for Transformations Between Equivalent Equations

- Adding (or subtracting) an equal quantity to (or from) both sides.
- Multiplying an equal, **non-zero** quantity on both sides.
- Dividing or multiplying by the reciprocal of a **non-zero** quantity on both sides.
- Applying arithmetic properties to rearrange expressions.
- Substitution of like relations.

Each of these keeps the solution set **invariant** or unchanging. Invariance is a *very* powerful property. (See the story of Emmy Noether, who used the properties of invariants to fundamentally change not only abstract algebra but also mathematical physics.)

We won't *prove* these, but we will provide general examples to justify them.

- Adding / subtracting the same quantity:

$$\begin{aligned} x + 15 &= 20 \\ (x + 15) - 15 &= 20 - 15 && \text{subtracting 15 from both sides} \\ x + (15 - 15) &= 5 && \text{associative property and evaluation} \\ x + 0 &= 5 && \text{additive inverse} \\ x &= 5 && \text{additive identity} \end{aligned}$$

  Sometimes we may add a *variable* to each side. For example:

$$\begin{aligned} x - y &= 5 - y \\ (x - y) + y &= (5 - y) + y \\ x &= 5 \end{aligned}$$

  demonstrates that the value of x does not depend on y.

- Multiplying (or dividing by) an equal, known **non-zero** quantity on both sides:

$$\begin{aligned} 5x &= 1 \\ \frac{1}{5} \cdot (5x) &= \frac{1}{5} \cdot 1 \\ (\frac{1}{5} \cdot 5) \cdot x &= \frac{1}{5} \\ 1 \cdot x &= \frac{1}{5} \\ x &= \frac{1}{5} \end{aligned}$$

  In this case, we are working with a constant $5 \neq 0$, so taking the reciprocal (or dividing) is well-defined.

- Multiplying (or dividing by) an equal, **unknown** quantity on both sides. We must take care not to divide by zero. Consider

$$xy = 1.$$

  Here, we **know** that $y \neq 0$ and $x \neq 0$, otherwise $xy = 0$. So here it is safe to move y over as in

$$x = \frac{1}{y}.$$

  But in

$$xy = z$$

  for some **unknown** z, we cannot say if x or y are zero!

- Arithmetic manipulations. Evaluating expressions and applying associative, commutative, and distributive properties to simplify the expression. Commonly referred to as "collecting like terms," or gathering all coefficients of the same variable.

$$\begin{aligned} 2 + (3 + x) - 7 + 2(x + y) &= -2 + 3x + 2y \\ (x^2 + 11) + x(3x + 4) &= 4x^2 + 4x + 11 \end{aligned}$$

- Substituting like quantities. We used this in the example above to reduce from a two variable system,

$$\begin{aligned} Y + N &= 100, \text{ and} \\ N &= Y + 4, \end{aligned}$$

  to the single equation $2Y + 4 = 100$.

37.7 Transformation Examples

See the text's Examples 1, 2, and 3 in Section 7.1. For example 3, you can add the fractions directly without multiplying by the common denominator as well. Multiplying the second fraction by $\frac{3}{3}$ gives

$$\frac{(x + 7) + 3(2x - 8)}{6} = \frac{7x - 17}{6} = -4.$$

37.8 Manipulating Formulæ by Transformations

Consider an equation for the perimeter of a rectangle, $P = 2L + 2W$. When you need to compute one variable from the others, treat the variables you know as numbers. To compute the length L given the perimeter P and width W, classify

- P and W as *known* variables, and
- L as the *unknown* variable.

To find a formula for L, treat P and W as if they were numbers and solve for L.

$$\begin{aligned} 2L + 2W &= P, & \\ 2L &= P - 2W & \text{by subtracting } 2W \text{ from both sides, and} \\ L &= \frac{P - 2W}{2} & \text{by dividing by the non-zero constant two.} \end{aligned}$$

Chapter 38

Homework for the twelfth week

38.1 Homework

Notes also available as PDF.

Practice is absolutely critical in this class.

Groups are fine, turn in your own work. Homework is due in or before class on Mondays.

- Exercises for 7.1:
  - 1, 2, 3, 4
  - 9, 10, 17, 18, 25, 26, 36, 37
  - Assume no variable is zero: 61, 62, 68, 69
  - 76
- Exercises for 7.2:
  - 21, 24, 26
  - 43 (Example 5 is six pages before this problem in my text)
- **Exercises for 8.2: delayed until next week**

Note that you *may* email homework. However, I don't use Microsoft™ products (*e.g.* Word), and software packages are notoriously finicky about translating mathematics.

If you're typing it (which I advise just for practice in whatever tools you use), you likely want to turn in a printout. If you do want to email your submission, please produce a PDF or PostScript document.

Chapter 39

Solutions for twelfth week's assignments

Also available as PDF.

39.1 Exercises for 7.1

Problem 1 **A** and **C** are linear.

Problem 2 For **B**, there is an exponent of two (x^2). And for **D**, there is an exponent of negative one ($\frac{1}{x}$).

Problem 3 Substituting, $3(6+4) = 3 \cdot 10 = 30$ on the left, and $5 \cdot 6 = 30$ on the right. Hence 6 **is a solution**.

Problem 4 Without evaluating the expressions, we can see that substituting -2 on the left yields an *even* number and that substituting -2 on the right yields an *odd* number. **Thus -2 cannot be a solution.**

But if we want to evaluate the expressions anyways, we have $5(-2+4) - 3(-2+6) = 5 \cdot 2 - 3 \cdot 3 = 10 - 12 = -2$ on the left, and $9(-2+1) = 9 \cdot -1 = -9$. Because $-2 \neq -9$, -2 is not a solution.

Problem 9 $7k + 8 = 1 \Rightarrow 7k = -7 \Rightarrow k = -1$.

Problem 10 $5m - 4 = 21 \Rightarrow 5m = 25 \Rightarrow m = 5$.

Problem 17 $2(x+3) = -4(x+1) \Rightarrow 2x + 6 = -4x - 4 \Rightarrow 6x = -10 \rightarrow x = \frac{-10}{6} = \frac{-5}{3}$.

Problem 18 $4(x-9) = 8(x+3) \Rightarrow 4x - 36 = 8x + 24 \Rightarrow -60 = 4x \Rightarrow x = -15$.

Problem 25 $-[2z-(5z+2)] = 2+(2z+7) \Rightarrow -2z+5z+2 = 9+2z \Rightarrow 3z+2 = 9+2z \Rightarrow z = 7$.

Problem 26 $-[6x-(4x+8)] = 9+(6x+3) \Rightarrow -6x+4x+8 = 12+6x \Rightarrow -2x+8 = 12+6x \Rightarrow -8x = 4 \Rightarrow x = \frac{-1}{2}$.

Problem 36 $\frac{3x}{4}+\frac{5x}{2} = 13 \Rightarrow \frac{3x+10x}{4} = 13 \Rightarrow \frac{13}{4}x = 13 \Rightarrow x = 4$.

Problem 37 $\frac{8x}{3}-\frac{2x}{4} = -13 \Rightarrow \frac{32x-6x}{12} = -13 \Rightarrow \frac{26}{12}x = -13 \Rightarrow x = \frac{-13\cdot 12}{13\cdot 2} = -6$.

Problem 61 $t = \frac{d}{r}$

Problem 62 $r = \frac{I}{pt}$

Problem 68 $r = \frac{C}{2\pi}$

Problem 69 $h = \frac{S-2\pi r^2}{2\pi r} = \frac{S}{2\pi r} - r$ (either is fine, the latter is better for calculators)

Problem 76

- In part a, $x = 93$. Then $y = .1\cdot 93 - 8.5 = 9.3 - 8.5 = .8$, or **800 000 tickets**.
- For part b, solve $.75 = .1x - 8.5$ for x. Then $x = (.75+8.5)/.1 = 9.25/.1 = 92.5$. **So the model predicts that the season spanning the latter half of 1992 through the first half of 1993 sold 750 000 tickets**. But reading into a model like this is tricky. I expect the authors intend the answer to be the 1992–1993 season.

39.2 Exercises for 7.2

Problem 21

1. Let x be the number of big-store shoppers, poor people.
2. Then $x-70 = y$, where y is the number of happy small-store shoppers[1].
3. $x+y = 442$, or $x+(x-70) = 442$.
4. From the above, $2x-70 = 442 \Rightarrow 2x = 512 \Rightarrow x = 256$.
5. So there are $x = 256$ big-store shoppers and $y = 256-70 = 186$ small-store shoppers.
6. The number of **big-store shoppers** was **70** more than the number of **small-store shoppers**, and the total number of these two bookstore types was $\mathbf{256+186 = 442}$.

Problem 24 Let W be the number of wins and L be the number of losses. Then $W = 3L-2$ and $W+L = 82$. So $3L-2+L = 82 \Rightarrow 4L = 84 \Rightarrow \mathbf{L = 21}$ is

[1]One source of local book stores is `http://www.librarything.com/local/`. Another is `http://www.indiebound.org/`.

the number of losses, and $\mathbf{W} = \mathbf{82} - \mathbf{21} = \mathbf{61}$ is the number of wins. This solution makes sense; both numbers are non-negative integers.

Problem 26 Let D be the number of votes for G.W. Bush, and S be the number of votes for A. Gore. Then $D + S = 537$ and $D = S + 5$. Then $2S + 5 = 537$, so $\mathbf{S} = \mathbf{266}$ and $\mathbf{D} = \mathbf{271}$. Again, this makes sense because the numbers are non-negative integers.

Problem 43

Percent	Investment	Interest
0.03	x	$0.03 \cdot x$
0.04	$12\,000 - x$	$0.04 \cdot (12\,000 - x)$
	$12\,000$	440

So $0.03 \cdot x + 0.04 \cdot (12\,000 - x) = 440$. Solving for x, $\mathbf{x} = \mathbf{4000}$.

He invested \$4 000 at 3% interest and \$8 000 at 4% interest. Checking, this totals to $0.03 \cdot 4000 + 0.04 \cdot 8000 = 120 + 320 = 440$.

Chapter 40

Homework for the thirteenth week

40.1 Homework

Notes also available as PDF.

Practice is absolutely critical in this class.

Groups are fine, turn in your own work. Homework is due in or before class on Mondays.

- Exercises for 8.2
  - 1, 2
  - 9, 10
  - 17, 18
  - 39
  - 57, 58
  - 63, 64
- Exercises for 8.3
  - 5, 6
  - 19, 20
  - 26, 28
  - 39, 40

  - 60, 64

- Exercises for 8.7
  - 3, 4
  - 15, 16
  - 24, 25
  - 35, 36
- Exercises for 8.8
  - 1, 2, 3, 4
  - 21, 22
  - 25, 26

Note that you *may* email homework. However, I don't use Microsoft™ products (*e.g.* Word), and software packages are notoriously finicky about translating mathematics.

If you're typing it (which I advise just for practice in whatever tools you use), you likely want to turn in a printout. If you do want to email your submission, please produce a PDF or PostScript document.

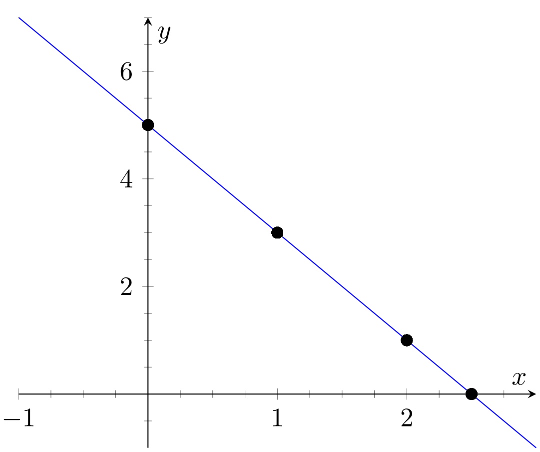
Chapter 41

Solutions for the thirteenth week's assignments

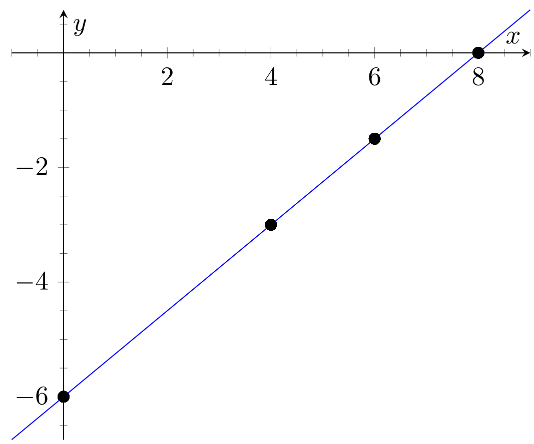
Also available as PDF.

41.1 Exercises for 8.2

Problem 1: $(0,5)$, $(\frac{5}{2},0)$, $(1,3)$, $(2,1)$



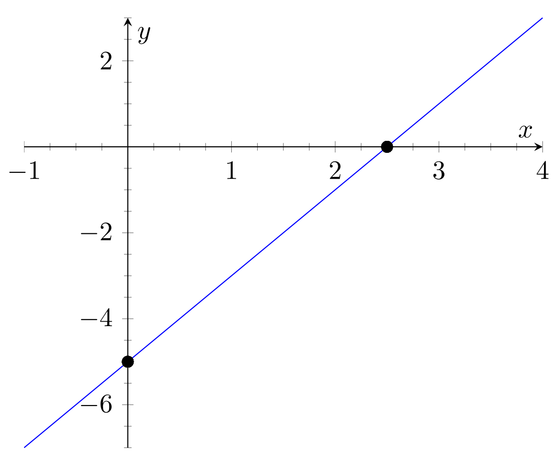
Problem 2: $(0,-6)$, $(8,0)$, $(6,\frac{-3}{2})$, $(4,-3)$



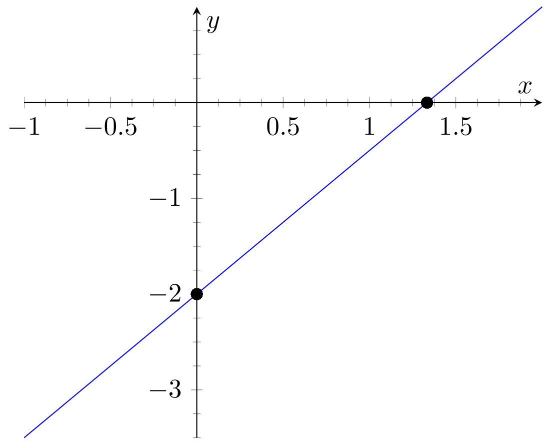
Problem 9: Substitute 0 for y, then solve for x. Or compute the intercept form.

Problem 10: Substitute 0 for x, then solve for y. Or compute the intercept form.

Problem 17: In intercept form: $\frac{x}{5/2} + \frac{y}{-5} = 1$, so the intercepts are $(\frac{5}{2}, 0)$ and $(0, -5)$.



Problem 18: In intercept form: $\frac{x}{4/3} + \frac{y}{-2} = 1$, so the intercepts are $(\frac{4}{3}, 0)$ and $(0, -2)$.



Problem 39: **Part a:** The two points are $(-1,-4)$ and $(3,2)$. Then $\Delta x = 3 - -1 = 4$, $\Delta y = 2 - -4 = 6$, so the slope is $\frac{\Delta y}{\Delta x} = \frac{6}{4} = \frac{3}{2}$.
Part b: The two points are $(1,-2)$ and $(-3,5)$. Then $\Delta x = -3 - 1 = -4$ and $\Delta y = 5 - -2 = 7$. The slope is $\frac{\Delta y}{\Delta x} = \frac{-7}{4}$.

Problem 57: L_1's slope is $\frac{7-6}{-8-4} = \frac{-1}{12}$. L_2's slope is $\frac{5-4}{-5-7} = \frac{-1}{12}$. The slopes are equal, so the lines are **parallel**.

Problem 58: L_1's slope is $\frac{12-15}{-7-9} = \frac{3}{16}$. L_2's slope is $\frac{5-8}{-20--4} = \frac{-3}{-16} = \frac{3}{16}$. These lines are **parallel**.

Problem 63: The "run", or change along the horizontal from the back of the deck to its fore, is $250 - 160 = 90$ feet. The "rise" is the drop from the back to the fore, or -63 feet. So the slope is $\frac{-63}{90} = \frac{-7}{10} = -0.7$. Note that if you start at the fore and face the aft, the slope will be negated (0.7). Both answers are correct so long as you explain the direction. In the end, this is a **70% grade**. Yes, that is steep.

Problem 64: A 13% grade is a slope of $\frac{13}{100}$, or 13 feet up for every 100 feet across. Given a run of 150 feet, the maximum rise is $\frac{13}{100} \cdot 150 = \frac{39}{2} = 19.5$ feet.

41.2 Exercises for 8.3

Problem 5: The slope is $\frac{0--3}{1-0} = 3$. With the y-intercept of -3, the slope-intercept form is $y = 3x - 3$.

Problem 6: The slope is $\frac{0--4}{2-0} = 2$. With the y-intercept of -4, the slope-intercept form is $y = 2x - 4$.

Problem 19: Starting with the form $y - y_0 = m(x - x_0)$, substituting gives $y - 8 = -2(x - 5)$. Solving for y and simplifying into slope-intercept form gives $y = -2x + 18$.

Problem 20: Starting with the form $y - y_0 = m(x - x_0)$, substituting gives $y - 10 = 1(x - 12)$. Solving for y and simplifying into slope-intercept form gives $y = x - 2$.

Problem 26: A slope of 0 is a horizontal line. Thus $y = -2$.

Problem 28: An undefined slope is a vertical line. Thus $x = -2$.

Problem 39: The points have the same y coordinate, so the line is horizontal. Thus $y = 5$ with a slope of zero.

Problem 40: The points have the same y coordinate, so the line is horizontal. Thus $y = 2$ with a slope of zero.

Problem 60: The slope of the latter line is $\frac{-2}{5}$, so $y - 1 = \frac{-2}{5}(x - 4)$ is the point-slope form. Reducing to slope-intercept form, $y = \frac{-2}{5}x + \frac{13}{5}$.

Problem 64: The slope of the latter line is $\frac{-5}{2}$. Thus a *perpendicular* line will have the slope $\frac{2}{5}$. In point-slope form, $y - -7 = \frac{2}{5}(x - 2)$, or $y = \frac{2}{5}x + \frac{-39}{5}$ in slope-intercept form.

41.3 Exercises for 8.7

Problem 3: $5 + 1 = 6$ and $5 - 1 = 4$, so $(5, 1)$ **is a solution**.

Problem 4: $8 - -9 = 8 + 9 = 17$ and $8 + -9 = 8 - 9 = -1$, so $(8, -9)$ **is a solution**.

Problem 15: Starting with the matrix of coefficients,

$$\begin{bmatrix} 7 & 2 & 6 \\ -14 & -4 & -12 \end{bmatrix},$$

adding twice the first row to the second produces

$$\begin{bmatrix} 7 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence both lines are the same, and the solution set consists of all points on the line $7x + 2y = 6$.

Problem 16: Here we start with

$$\begin{bmatrix} 1 & -4 & 2 \\ 4 & -16 & 8 \end{bmatrix}.$$

Subtracting four times the first row from the second yields

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Again, these lines are the same, and the solution set consists of all points on the line $x - 4y = 2$.

Problem 24: One method starts by solving the first equation for $y = 3x - 5$. Substituting into the second, $x + 2(3x - 5) = 0$ gives $7x - 10 = 0$ or $x = \frac{10}{7}$. Now substituting this x into $y = 3x - 5$ yields the solution point $(\frac{10}{7}, \frac{-5}{7})$.

Problem 25: The second equation provides $y = 2x - 1$. Then the first becomes $-x - 4(2x - 1) = -14$, or $-9x + 4 = -14$, giving $x = 2$. Then $y = 3$ and the solution is $(2, 3)$

Problem 35: Start by writing out the coefficients:

$$\begin{bmatrix} 3 & 2 & 1 & 8 \\ 2 & -3 & 2 & -16 \\ 1 & 4 & -1 & 20 \end{bmatrix}$$

Now rearrange to make cancellation easier by hand:

$$\begin{bmatrix} 1 & 4 & -1 & 20 \\ 2 & -3 & 2 & -16 \\ 3 & 2 & 1 & 8 \end{bmatrix}$$

(**Note:** If using a computer or calculator, you really should rearrange so you always divide by the largest magnitude entry remaining in the column. Here, I would not have altered the order. But this happens to be all-integer if you chose the correct operations.)
Now subtract the first row from the two remaining rows:

$$\begin{bmatrix} 1 & 4 & -1 & 20 \\ 0 & -11 & 4 & -56 \\ 0 & -10 & 4 & -52 \end{bmatrix}$$

Next, recognize that $-10 - -11 = 1$ and subtract the second row from the last row:

$$\begin{bmatrix} 1 & 4 & -1 & 20 \\ 0 & -11 & 4 & -56 \\ 0 & 1 & 0 & 4 \end{bmatrix}$$

Swap rows to place the second column's 1 in the second row:

$$\begin{bmatrix} 1 & 4 & -1 & 20 \\ 0 & 1 & 0 & 4 \\ 0 & -11 & 4 & -56 \end{bmatrix}$$

Add -11 times the second row to the last:

$$\begin{bmatrix} 1 & 4 & -1 & 20 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 4 & -12 \end{bmatrix}$$

Now divide the last by four:

$$\begin{bmatrix} 1 & 4 & -1 & 20 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Eliminate along the last column by adding the last row to the first:

$$\begin{bmatrix} 1 & 4 & 0 & 17 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

And eliminate the rest of the second column by subtracting four times the second row from the first:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

This gives the solution $(\mathbf{1}, \mathbf{4}, -\mathbf{3})$.

Problem 36: Begin with:

$$\begin{bmatrix} -3 & 1 & -1 & -10 \\ -4 & 2 & 3 & -1 \\ 2 & 3 & -2 & -5 \end{bmatrix}$$

Again, if you chose the order correctly, all arithmetic will be with integers. But this time I'll perform the operations in the sensible numerical order. First, swap the first two rows to place the largest magnitude entry, -4, at the top:

$$\begin{bmatrix} -4 & 2 & 3 & -1 \\ -3 & 1 & -1 & -10 \\ 2 & 3 & -2 & -5 \end{bmatrix}$$

Now divide the first row by four:

$$\begin{bmatrix} 1 & -0.5 & -0.75 & 0.25 \\ -3 & 1 & -1 & -10 \\ 2 & 3 & -2 & -5 \end{bmatrix}$$

Subtract multiples of the first row from the others (-3 for the second row, 2 for the third):

$$\begin{bmatrix} 1 & -0.5 & -0.75 & 0.25 \\ 0 & -0.5 & -3.25 & -9.25 \\ 0 & 4 & -0.5 & -5.5 \end{bmatrix}$$

Swap the second and third rows:

$$\begin{bmatrix} 1 & -0.5 & -0.75 & 0.25 \\ 0 & 4 & -0.5 & -5.5 \\ 0 & -0.5 & -3.25 & -9.25 \end{bmatrix}$$

Divide the second row by four:

$$\begin{bmatrix} 1 & -0.5 & -0.75 & 0.25 \\ 0 & 1 & -0.125 & -1.375 \\ 0 & -0.5 & -3.25 & -9.25 \end{bmatrix}$$

Subtract -0.5 times the second row from the third:

$$\begin{bmatrix} 1 & -0.5 & -0.75 & 0.25 \\ 0 & 1 & -0.125 & -1.375 \\ 0 & 0 & -3.3125 & -9.9375 \end{bmatrix}$$

Divide the third row by -3.3125:

$$\begin{bmatrix} 1 & -0.5 & -0.75 & 0.25 \\ 0 & 1 & -0.125 & -1.375 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Subtract -0.125 times the third row from the second:

$$\begin{bmatrix} 1 & -0.5 & -0.75 & 0.25 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Now subtract -0.5 times the second row and -0.75 times the third row from the first:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This gives the solution $(2, -1, 3)$.

Note: Solving that last problem in an environment like Octave is slightly easier:

```
octave> [-3 1 -1; -4 2 3; 2 3 -2] \ [-10; -1; -5]
ans =

   2
  -1
   3
```

There is a **lot** of work in making that magic \ operator function correctly.

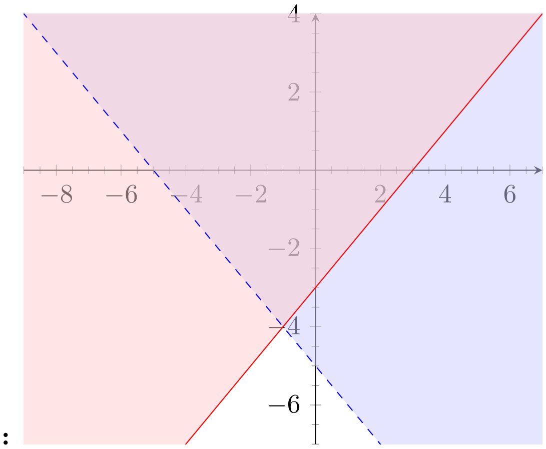
41.4 Exercises for 8.8

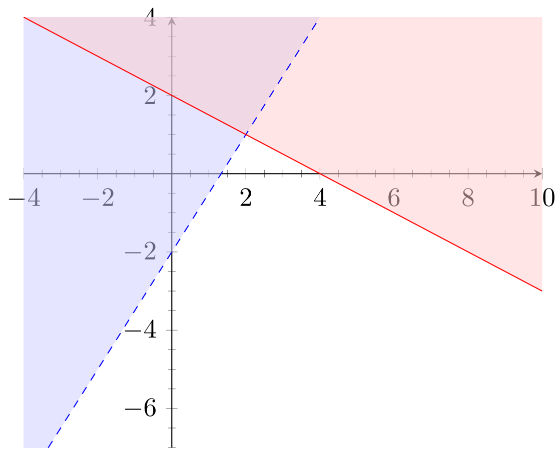
Problem 1: C: Dashed lines denote $<$ and $>$.

Problem 2: A: Solid lines denote $\leq$ and $\geq$.

Problem 3: B

Problem 4: D

Problem 21: 

Problem 22: 

Problem 25: The interesting vertices are the x- and y-intercepts along with the intersections of those lines, along with the origin $(0, 0)$. The intersection is at $(1.2, 1.2)$. All values of $5x + 2y$ at these points:

Point	Value
$(0, 0)$	0
$(0, 2)$	4
$(3, 0)$	15
$(0, 6)$	12
$(1.5, 0)$	7.5
$(1.2, 1.2)$	8.4

We could draw the graph to determine which vertices are in the feasible region. Alternately, we can test points against all the inequalities, beginning with the largest value and working downwards. The point $(3, 0)$ does not satisfy $4x + y \leq 6$ and is not feasible. The point $(0, 6)$ does not satisfy $2x + 3y \leq 6$ and is not feasible. The point $(1.2, 1.2)$ satisfies all the inequalities, so the **largest value is 8.4 occuring at** $(\mathbf{1.2}, \mathbf{1.2})$.

Problem 26: Listing all of the interesting points only once gives

Point	Value
$(0, 0)$	0
$(10, 0)$	10
$(4, 0)$	4
$(0, 10)$	30
$(20/3, 10/3)$	50/3
$(10/3, 5/3)$	25/3

Now check points from the least value upwards. $(0, 0)$ does not satisfy $5x + 2y \geq 20$. $(4, 0)$ does not satisfy $2y \geq x$. $(10/3, 5/3)$ satisfies every constraint. Thus **the minimal value is 25/3 at point** $(\mathbf{10/3}, \mathbf{5/3})$**.**

Chapter 42

Homework for the fourteenth week

42.1 Homework

Notes also available as PDF.

Practice is absolutely critical in this class.

Groups are fine, turn in your own work. Homework is due in or before class on Mondays.

- Exercises for 7.3
  - 25, 26, 68, 70
- Exercises for 7.4
  - 64, 65, 66 (note: making a profit implies $R - C > 0$ where R is the revenue and C is the cost)
- Exercises in 7.5
  - 60, two different ways. First, substitute points into $x^2+(x+30)^2-150^2$ and plot the line segments. Try $x \in \{80, 85, 90, 95, 100\}$. In this case, you'll happen to find the answer. For the other way, use the Pythagorean theorm as in the text.
- Exercises in 8.1
  - 56: Use the point formula for a line,

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0},$$

to determine the equation of the closest points to each requested x. Then substitute the x value in the middle and find the y.

- Exercises in 8.3:
  - 70, 72, 74
- Exercises in 8.6:
  - 50
- Exercises in 8.7:
  - Use either substitution or elimination: 50, 78 (yes, I had to assign a "speed of a train leaving..." problem), 86
- Exercises in 8.8:
  - 30, 34

Note that you *may* email homework. However, I don't use Microsoft™ products (*e.g.* Word), and software packages are notoriously finicky about translating mathematics.

If you're typing it (which I advise just for practice in whatever tools you use), you likely want to turn in a printout. If you do want to email your submission, please produce a PDF or PostScript document.

Chapter 43

Solutions for the fourteenth week's assignments

Also available as PDF.

43.1 Exercises for 7.3

Problem 25 The ratio is $\frac{2.5 \text{ oz oil}}{1 \text{ gal gas}}$. So given 2.75 gallons, $\frac{2.5 \text{ oz oil}}{1 \text{ gal gas}} \cdot 2.75 \text{ gal gas} =$ **6.875 ounces** of oil are required.

Problem 26 Here the ratio is $\frac{5.5 \text{ oz oil}}{1 \text{ gal gas}}$. Given 22 ounces of oil, $\frac{1 \text{ gal gas}}{5.5 \text{ oz oil}} \cdot 22 \text{ oz oil} =$ **4 gallons** of gas are required.

Problem 68 "Varies directly" implies a proportional relationship. Here the ratio is $\frac{5 \text{ psi}}{200 \text{ deg K}}$. So at 300 degrees Kelvin, the pressure is $\frac{5 \text{ psi}}{200 \text{ deg K}} \cdot 300 \text{ deg K} =$ **7.5 psi**.

Problem 70 The correct ratio here is $\frac{12 \text{ pounds}}{3 \text{ in}}$, and the force to compress 5 inches is $\frac{12 \text{ pounds}}{3 \text{ in}} \cdot 5 \text{ in} =$ **20 pounds**.

43.2 Exercises for 7.4

Problem 64 The target heart rate for a 35 year old lies in $[129.5, 157.25]$. Rounding this to nearest **even** (safer in general) gives the interval $[\mathbf{130}, \mathbf{157}]$. When dealing with intervals like these, however, rounding each endpoint to nearest is not necessarily the best idea. In this case, the result was *within* the original interval, so all numbers in the rounded interval still satisfy the relationship. If we had rounded 129.5 to 129, then the lower endpoint would not have satisfied the relationship. Which direction to round depends on the problem and assumptions in the model[1]. **Your age:** Likely is less than mine, which in turn is less than the first part. Note that you can treat each side, $.7(220 - A)$ and $.85(220 - A)$, as a line. Both have negative slopes, so both *decrease* with increasing age.

Problem 65 At break-even, the cost C is equal to the revenue R. So at break-even, $20x + 100 = 24x$ and $x = 25$. Now the question becomes on which side the company shows a *profit*, or $R - C > 0$. Substituting for R and C, $(24x) - (20x + 100) > 0$ or $x > 25$. The smallest whole number of units x to show a profit is then **26** and *not* 25.

Problem 66 Here $R = 5.5x$ and $C = 3x + 2300$, so $R - C > 0$ becomes $5.5x - (3x + 2300) > 0$ or $2.5x > 2300$ and $x > 920$. So the smallest profitable x is **921**.

43.3 Exercises for 7.5

When what I say clearly does *not* apply to the problem, you should tell me. Here I assigned Problem 60 in Section 7.5 when I meant Section 7.7. I've included the answer for the problem in Section 7.5, even though it's a pointless problem.

Problem 60 $(p^{-1})^3 p^{-4} = p^{-3} p^{-4} = \mathbf{p^{-7}}$

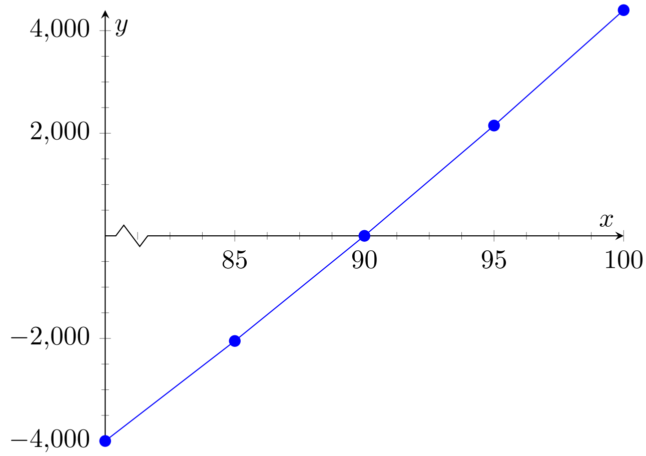
43.4 Exercises for 7.7

Problem 60 by plotting The function to plot is $y = x^2 + (x + 30)^2 - 150^2 = 2x^2 + 60x - 21600$. Evaluating at the points $x \in \{80, 85, 90, 95, 100\}$ gives:

x	80	85	90	95	100
y	-4000	-2050	0	2150	4400

So the zero is at $x = 90$. Plotting these segments:

[1] See the IEEE interval standardization group at `http://www.cs.utep.edu/interval-comp/standard.html` for links to more information.



Problem 60 by Pythagorean thm By the Pythagorean theorem, $x^2 + (x+30)^2 = 150^2$. Then we need to find roots of $2x^2 + 60x - 21600 = 2(x^2 + 30x - 10800)$. We can solve this by simply trying points or by the quadratic equation. For the latter, we can use the simpler $x^2 + 30x - 10800$ and find

$$\begin{aligned} x &= \frac{-30 \pm \sqrt{30^2 + 4 \cdot 10800}}{2} \\ &= \frac{-30 \pm 210}{2} = -15 \pm 105 \\ &= 90 \text{ or } -120. \end{aligned}$$

For using points, find convenient numbers in the problem and try them. Here, to numbers that pop out are 30 and 150. Evaluating at each gives $(30, -14400)$ and $(150, 50400)$, so we know the zero must be somewhere between them. Half-way is $(30 + 150)/2 = 90$, so trying 90 finds the solution.

43.5 Exercises for 8.1

Problem 56, using the point form of the line The two points closest to 1985 are $(1980, 4.5)$ and $(1990, 5.2)$. The point form here is

$$\frac{x - 1980}{1990 - 1980} = \frac{y - 4.5}{5.2 - 4.5}.$$

Plugging in $x = 1985$ and solving for y gives $y = \mathbf{4.85}$ **million**. Repeating around 1995,

$$\frac{x - 1990}{2000 - 1990} = \frac{y - 5.2}{5.8 - 5.2}.$$

Using $x = 1995$ gives $y = \mathbf{5.5}$ **million**.

43.6 Exercises for 8.3

Problem 70 *(This was the "bad graph" example I used previously to show how a linear relationship can be hidden.* We can start with the point form,

$$\frac{x-150}{1400-150} = \frac{y-5000}{24000-5000}.$$

Solving for y gives the slope-intercept form,

$$y = 15.2x + 2720.$$

To verify this, check that $x = 150$ gives $y = 5000$, and $x = 1400$ gives $y = 24000$.

Problem 72 Part a Again, starting from point form is easiest:

$$\frac{x-5}{7-5} = \frac{y-24075}{26628-24075}.$$

Solving for y,

$$\begin{aligned} y &= 24075 + \frac{2553}{2}(x-5) \\ &= 1276.5x + 17692.5. \end{aligned}$$

Part b The slope from 1995 to 1997 is positive, but the graph shows a negative slope from 1993 or 1994 to 1995. Thus the linear model above will not approximate those well. The graph shows non-linear variation, so there is no reason to expect the 1995–1997 line to continue to 1998, so **no** to all.

Problem 74 Part a and b The points in question are $(\mathbf{0}, \mathbf{32})$ and $(\mathbf{100}, \mathbf{212})$.

Part c The slope is $\frac{\Delta y}{\Delta x} = \frac{212°\text{F}-32°\text{F}}{100°\text{C}-0°\text{C}} = \frac{\mathbf{9}°\text{F}}{\mathbf{5}°\text{C}}$.

Part d The point $(0, 32)$ provides the y-intercept, so the line is $\mathbf{y} = \frac{\mathbf{9}°\text{F}}{\mathbf{5}°\text{C}}\mathbf{x} + \mathbf{32}°\text{F}$.

Part e A function of x in terms of y: $\mathbf{x} = \frac{\mathbf{5}°\text{C}}{\mathbf{9}°\text{F}}(\mathbf{y} - \mathbf{32}°\text{F})$.

Part f The graph shows that 50°C is 122°F. You can tell that the graph is of the conversion °C → °F because the y-intercept is positive.

43.7 Exercises for 8.6

Problem 50 At 7% compounded quarterly[2], \$60 000 will grow to in \$60 000 $\cdot (1 + \frac{0.07}{4})^{20} \approx$ \$84886.69 5 years. Compounded "continuously"[3], the same

[2]Note that the text's definition of "compounded quarterly" is not used by all financial institutions. Always check how your institution defines their terms. Also, intermediate quantities in the calculation may be rounded according to local laws and the institution's rules. Yes, it really is this complicated.

[3]Again, check with individual institutions about their definitions.

amount will grow to $\$60\,000 \cdot e^{0.0675 \cdot 5} \approx \$84086.38$. Here the higher rate compounded quarterly is the better one, with a difference of *approximately* \$800.32.

43.8 Exercises for 8.7

Problem 50 The nonsensical system here is $W = L - 44$ ft and $2L + 2W = 288$ ft. Solving by substituting the former into the latter gives $L = 144\text{ ft} - W = 144\text{ ft} - L + 44$ ft, so $L = \mathbf{94\ ft}$. Now $W = L - 44\text{ ft} = \mathbf{50\ ft}$.

Problem 78 The first statement gives $\frac{150\text{ km}}{T\text{ km/hr}} = \frac{400\text{ km}}{P\text{ km/hr}}$ or equivalently $150\text{ km} \cdot P\text{ km/hr} = 400\text{ km} \cdot T\text{ km/hr}$. The second statement gives $P\text{ km/hr} = 3T\text{ km/hr} - 20\text{ km/hr}$. Solving, the plane's speed is T km/hr = **60 km/hr**, and the train's speed is P km/hr = **160 km/hr**. Plugging these into either of the given equations verifies the solution.

Problem 86 Translating into algebra,

$$C = A + B + 10,$$
$$B = 2A, \text{ and}$$
$$A + B + C = 490.$$

This is best solved by substitution. Written in terms of C, the last equation becomes $(C - 10) + C = 490$, or $\mathbf{C = 250}$. Now writing the first in terms of A after substituting C, $250 = A + 2A + 10$ or $\mathbf{A = 80}$. Now $\mathbf{B = 160}$. Substituting these into the equations above verifies this solution.

43.9 Exercises for 8.8

Problem 30 Let A and B be the servings of each product. Summarizing the information:

Product	I (g/serving)	II (g/serving)	cost (\$/serving)
A	3	2	0.25
B	2	4	0.40

So the function to minimize is the cost $0.25A + 0.40B$. The constraints are that there be at least 15 g of I, or $3A + 2B \geq 15$, and 15 g of II, or $2A + 4B \geq 15$. There also are trivial constraints $A \geq 0$ and $B \geq 0$. The problem to solve is to

$$\begin{aligned}
\min_{A,B} \quad & 0.25A + 0.40B \\
\text{subject to} \quad & 3A + 2B \geq 15, \\
& 2A + 4B \geq 15, \\
& A \geq 0, \text{ and} \\
& B \geq 0.
\end{aligned}$$

The first two lines intersect at $(3.75, 1.875)$. The A intercepts are 5 and 7.5, and only $(0, 7.5)$ is feasible. The B intercepts are 7.5 and 3.75, and only $(7.5, 0)$ is feasible. So the points to check and the function values are

Point	Cost
$(3.75, 1.875)$	\$1.6875
$(0, 7.5)$	\$3
$(7.5, 0)$	\$1.875

So the cheapest combination is 3.75 of A and 1.875 of B at \$1.6875.

Problem 34 Summarizing and assigning variables:

Kind	Variable	Oven time (hr)	Decorating time (hr)	Profit (\$)
Cookie	C	1.5	$\frac{2}{3}$	20
Cake	A	2	3	30

So the problem is to

$$\begin{aligned}
\max_{C,A} \quad & 20C + 30A \\
\text{subject to} \quad & 1.5C + 2A \leq 15, \\
& \frac{2}{3}C + 3A \leq 13, \\
& C \geq 0, \text{ and} \\
& A \geq 0.
\end{aligned}$$

The interesting points and their values:

Point	Profit (\$)
$(0, 0)$	0
$(6, 3)$	210
$(0, \frac{13}{3})$	130
$(10, 0)$	200

So the best combination is **6 cookie batches and 3 cake batches** for a **profit of \$210**.

Chapter 44

Third exam, *due 1 December*

Available as PDF. Remember, this is due on **1 December, 2008**.

Chapter 45

Third exam solutions

Available as PDF.

Chapter 46

Final exam

Available as PDF.

Part V

Resources

Chapter 47

Math Lab

See the Math Lab Information Sheet for details.

Room 209 of the J. F. Hicks Memorial Library. Tutoring and additional material.
Run by Prof. Charlotte Ingram.

Chapter 48

On-line

As with all things, question the provenance of on-line resources before relying upon them. This list is not comprehensive and does not provide endorsements; this list is just a starting point.

48.1 General mathematics education resources

Encyclopedia:

- Planet Math
- Wolfram Mathworld

Texts:

- Wikibooks
- George Cain's list of online mathematics textbooks
- Alex Stef(?)'s list of texts

48.2 Useful software and applications

This list is for future reference. Each item has a somewhat steep learning curve that is outside our scope. These may not be immediately useful for this course, but they can be useful for playing with ideas quickly.

Exploratory and programming environments:

- Linear algebra: Octave
- Statistics: R

- Geometry: Geomview
- Algebra: Maxima, YACAS, others...
- Spreadsheet: OpenOffice, SIAG, others... Note: spreadsheets often are made notorious for their poor quality arithmetic.