Math 131 Test

Sequences, Problem Solving, Sets, and Logic

19 September, 2008

Ten questions, each worth the same amount. Complete ***six*** of your choice. I will only grade the first six I see. Make sure your name is on the top of each page you return.

Explain your reasoning for each problem whenever appropriate; that helps me give partial credit. Perform scratch work on scratch paper; keep your explanations clean. Make final answers obvious by boxing or circling them.

And remember to read and answer the entire question.

1 Inductive and deductive reasoning

Pascal's triangle is a sequence of rows where each row is formed by adding pairs of numbers from the previous row. Written in rather boring table form, each entry is the sum of the entry directly above and above to the left:

row #					
0	1				
1	1	1			
2	1	2	1		
3	1	3	3	1	
4	$1 = \mathbf{0} + 1$	$4 = 1 + 3$	$6 = 3 + 3$	$4 = 3 + 1$	$1 = 1 + \mathbf{0}$

If the entry lies off the side of the table, we let it be zero as in the bold-faced terms. So the entry $P_{i,j} = P_{i-1,j-1} + P_{i-1,j}$, where i is the row and j is the column. For example, entry $P_{4,1} = P_{3,0} + P_{3,1} = \mathbf{0} + 1 = 1$, and entry $P_{4,3} = P_{3,2} + P_{3,3} = 3 + 3 = 6$.

- Reasoning **inductively**, explain the pattern in the second column. Provide both the pattern and *why* you think it is true.
- Reasoning **deductively**, explain the same pattern using the entries in the first column. Again, explain *why* in a deductive, more formal manner. Use the formula for computing each entry of $P_{i,2}$.
- Reasoning **inductively** again, examine the sum of the entries in each row. What is the pattern? Explain your reasoning.

2 Successive differences

Use successive differences to find the next two terms of the following sequence:

i	1	2	3	4	5	$\cdots$
A_i	4	11	22	37	56	$\cdots$

Again using successive differences, what are the next two terms of:

i	1	2	3	4	5	$\cdots$
B_i	3	7	13	21	31	$\cdots$

Reasoning inductively, find a simple formula for $g(i) = A_i - B_i$, the difference between the two sequences:

i	1	2	3	4	5	$\cdots$
$A_i - B_i$	4-3	11-7	22-13	37-21	56-31	$\cdots$

Find a simple formula for the remaining $h(i) = B_i - (A_i - B_i)$:

i	1	2	3	4	5	$\cdots$
$B_i - (A_i - B_i)$	3-(4-3)	7-(11-7)	13-(22-13)	21-(37-21)	31-(56-31)	$\cdots$

Now you have $g(i) = A_i - B_i$ and $h(i) = B_i - (A_i - B_i) = B_i - g(i)$. Derive a formula for the original B_i. Use it to verify the sixth and seventh terms of B_i.

3 Problem solving: searching

This problem is from G. Pólya's "How to Solve It".

Among Grandfather's papers a bill was found:

72 turkeys $ _67.9 _

The first and last digits of the number that obviously represented the total prive of those fowls are replaced here by blanks, for they have faded and are now illegible.

What are the two faded digits and what was the price of one turkey? Explain your reasoning.

Hint: The total is less than $500 and greater than $300. The price per turkey is not a whole number of dollars, but it is a whole number of cents.

Structure your explanation according to Pólya's principles: **understanding the problem**, **devising a plan**, **carrying out the plan**, and **looking back**.

4 Problem solving: finding and following dependencies

You are watching the line form in front of a bank teller before the teller opens. The line forms and is served in order of arrival.

- Bill arrives 10 minutes after Joan.
- Robert held the door open for Morgan and thus arrived after Morgan.
- Chris rushed in the door shortly before Morgan and Robert arrived, and no one else arrived between Robert, Morgan, and Chris.
- Chris thankful to see only one person in line.

Assuming that these are all the people who arrived, what is the order in which the people are served? Explain how you arrive at your answer.

Structure your explanation according to Pólya's principles: **understanding the problem**, **devising a plan**, **carrying out the plan**, and **looking back**.

5 Problem solving: patterns

What is the units digit (last digit) of 3^{25}? Using the pattern found for 3^k, find the units digit of 9^{25}, 27^{25}, and 81^{25}? Remember that $9 = 3^2$, $27 = 3^3$, and $81 = 3^4$.

Now find the units digit of 5^{25}. Given the units digit of 3^{25} and 5^{25}, what is the units digit of 15^{25}?

Explain your reasoning throughout. Structure your explanation according to Pólya's principles: **understanding the problem**, **devising a plan**, **carrying out the plan**, and **looking back**.

6 Set theory: definitions and relations

- Write out the set $\{2x+1 \mid x \in \mathbb{J}, -3 < x \leq 2\}$ by listing all its elements in set notation.
- Write the following set succinctly in set-builder notation: $\{7, 10, 13, 16, 19, 22\}$.
- Given two sets A and B, when is $A \subset B$?
- Given two sets A and B, when is $A \supset B$?
- What is a *proper* subset?
- What are two ways to write an empty set symbolically?

Fill in each with the most appropriate relation ($\in$, $\subset$, $\subsetneq$, or no relation at all):

- $1 \_\_ \{x/3 \mid x \text{ is a whole number less than five}\}$
- $\{\emptyset\} \_\_ A$ for all sets A
- $6 \_\_ \{1, 2, 3, 4, 5, 6\}$
- $\{6\} \_\_ \{1, 2, 3, 4, 5, 6\}$

Under what circumstances are each of the following true for sets A and B?

- $A \cup B = A$
- $A \setminus B = B$
- $A \subset B$ for any set B

7 Set theory: operations and relations

Define the result of the following operations using set-builder notation and symbolic logic:

- $A \cup B$
- $A \setminus B$
- $(A \setminus B) \cup C$

Draw two Venn diagrams for each of the following illustrating different ways A and B can be related (*e.g.* one is a subset of another, all sets are distinct, or other possibilities):

- $A \cap B$
- $A \setminus B$
- $(A \setminus C) \cap (B \setminus C)$

8 Logic: truth tables

Write truth tables defining the three operations $\neg$, $\wedge$, and $\vee$.

Use truth tables to determine if these statements are tautologies. State clearly if each is a tautology or not.

- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- $\neg p \vee \neg q \equiv \neg(p \vee q)$

What logical expression fits the following truth table? Verify the expression by constructing a truth table. Do you need all eight lines to verify your expression?

p	q	r	$f(p,q,r)$
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	0
0	0	1	1
0	0	0	1

9 Logic: negations and conditionals

- Use De Morgan's laws and properties of logic to show that

  $$p \vee q \vee \neg(p \wedge q)$$

  is a tautology. Work symbolically rather than building a truth table.
- Remembering that $\models p \rightarrow q \equiv \neg p \vee q$, is $q \wedge \neg(p \rightarrow q)$ ever true? Derive the result symbolically.
- Combine $\models p \rightarrow q \equiv \neg p \vee q$ with De Morgan's laws to write the set different $A \setminus B = \{x \mid x \in A \wedge \neg(x \in B)\}$ using the logical $\rightarrow$ operator.
- In words, describe the cases when $p \rightarrow q$ is true.
- The logical definition of the set expression $A \subset B$ is $x \in A \rightarrow x \in B$. Use the **contrapositive** to show that the empty set is a subset of all sets.

10 Logic: quantifiers

- Translate into a symbolic statement:

  Some student answered this question.

  What is the quantifier? What is the property or predicate?

- What is the logical negation of the statement:

  All students finished the test early.

  Express the negation both symbolically and in English.

- Express the following statement and its negation symbolically:

  Almost every student is confident about answering some question on this test.