Math 131 Test and Solutions

Algebra, Lines, and Models

24 November, 2008

Due in class on 1 December, 2008

Following are eight questions, each worth the same amount. Complete ***six*** of your choice. I will only grade the first six I see. Make sure your name is on the top of each page you return.

Explain your reasoning for each problem whenever appropriate; that helps me give partial credit. Perform scratch work on scratch paper; keep your explanations clean.

Make final answers obvious by boxing or circling them. When a question asks you to construct a table or perform a computation, showing the table or writing out the computation's steps is a part of the question and is **not optional**.

And remember to read and answer the entire question. There is copious explanation before a few problems. The explanation repeats some relevant material from class.

This exam also will be posted at

`http://jriedy.users.sonic.net/VI/math131-f08/`.

Mail me at jason@acm.org with any questions.

Contents

1 Interpolating From a Table

The US Naval Observatory publishes a table of sunrise and sunset times for any location worldwide and any year at `http://aa.usno.navy.mil/data/docs/RS_OneYear.php`. The following data is for 2009 in Bristol, VA. The times ignore daylight savings time.

Date	Sunrise	Sunset
1 January	7:41am	5:24pm
1 February	7:30am	5:55pm
1 March	6:59am	6:23pm
1 April	6:14am	6:51pm
1 May	5:35am	7:17pm
1 June	5:12am	7:42pm

Estimate the following times by interpolating between the closest dates in the above table:

1. Sunrise on 14 February
   Solution: With the time in minutes, the nearest points are 1 February's $(32, 450)$ and 1 March's $(60, 419)$. The point we are looking for is at the day number 45. Substituting into the point form,
   $$\frac{45-32}{60-32} = \frac{T-450}{419-450}.$$
   Solving gives $T \approx 435.61$, or just before **7:16am**.

2. Sunset on 28 January
   Solution: With the time in minutes, the nearest points are 1 January's $(1, 1044)$ and 1 February's $(32, 1075)$. The point we are looking for is at the day number 28. Substituting into the point form,
   $$\frac{28-1}{32-1} = \frac{T-1044}{1075-1044}.$$
   Solving gives $T = 1071$, or at **5:51pm**.

3. Sunset on 28 May
   Solution: With the time in minutes, the nearest points are 1 May's $(121, 1157)$ and 1 June's $(153, 1182)$. The point we are looking for is at the day number 148. Substituting into the point form,
   $$\frac{148-121}{153-121} = \frac{T-1157}{1182-1157}.$$
   Solving gives $T \approx 1159.42$, or just after **7:19pm**.

4. Sunrise on 15 April
   Solution: With the time in minutes, the nearest points are 1 April's $(91, 374)$ and 1 May's $(121, 335)$. The point we are looking for is at the day number 105. Substituting into the point form,
   $$\frac{105-91}{121-91} = \frac{T-374}{335-374}.$$
   Solving gives $T = 355.8$, or just before **5:56am**.

To interpolate, treat the closest two sunrise or sunset times as points (day, time), where the day is the day of the year. So 1 January is day 1, 1 February is day 32, *etc.* Connect the two points by a line and derive a function for the time given the day (also known as the slope-intercept form of the line). Then evaluate that function on the required day.

Is a linear model reasonable? To test this, find the equations of the following lines:

1. connecting 1 January and 1 June,
2. connecting 1 February and 1 April, and
3. connecting 1 March and 1 May.

Are these lines nearly the same? Explain.

Solution: Let's test sunrise times and assume sunset times are similar.

Date range	1 Jan — 1 June	1 Feb — 1 Apr	1 Mar — 1 May
Point form	$\frac{D-1}{152-1}=\frac{T-461}{312-461}$	$\frac{D-32}{91-32}=\frac{T-450}{374-450}$	$\frac{D-60}{121-60}=\frac{T-419}{335-419}$
Slope-intercept	$T=\frac{-149}{151}D+\frac{69462}{151}$	$T=\frac{-76}{59}D+\frac{26518}{59}$	$T=\frac{-84}{61}D+\frac{25499}{61}$
Approx.	$T\approx-0.9868D+460.01$	$T\approx-1.2881D+449.46$	$T\approx-1.3770D+418.02$

The intercepts may be close, but the slopes are massively different. So a linear model likely is not the best.

The sunset times show a similar divergence:

Date range	1 Jan — 1 June	1 Feb — 1 Apr	1 Mar — 1 May
Point form	$\frac{D-1}{152-1}=\frac{T-1044}{1182-1044}$	$\frac{D-32}{91-32}=\frac{T-1075}{1131-1075}$	$\frac{D-60}{121-60}=\frac{T-1103}{1157-1103}$
Slope-int. approx.	$T\approx0.9139D+1043.1$	$T\approx0.9492D+1044.6$	$T\approx0.8852D+1049.9$

This makes sense. The Earth is a sphere with its axis of rotation tilted. As the Earth revolves around the Sun, that tilt makes the path of the sun a curve and not a line.

2 Forms of Lines

We discussed many forms of lines in class. Among them are the following, where (x_0, y_0) and (x_1, y_1) are points, m is the slope, x_{int} is the x-intercept, and y_{int} is the y-intercept:

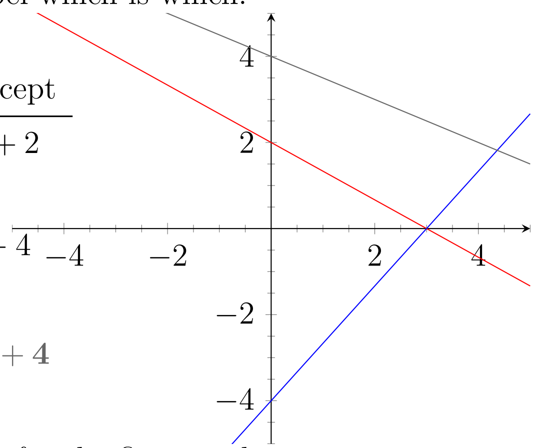
standard form: $ax + by = c$ slope-intercept form: $y = mx + y_{\text{int}}$

intercept form: $\dfrac{x}{x_{\text{int}}} + \dfrac{y}{y_{\text{int}}} = 1$ point form: $\dfrac{x - x_0}{x_1 - x_0} = \dfrac{y - y_0}{y_1 - y_0}$

parametric form: $x = x_0 + t \cdot (x_1 - x_0)$
$y = y_0 + t \cdot (y_1 - y_0)$

Fill out the rows of the following table by converting the given lines into the other forms, and then **plot the lines** and label which is which:

standard	intercept	slope-intercept
$\mathbf{2x + 3y = 6}$	$\frac{x}{3} + \frac{y}{2} = 1$	$y = \frac{-2}{3}x + 2$
$-4x + 3y = -12$	$\mathbf{\frac{x}{3} + \frac{y}{-4} = 1}$	$y = \frac{4}{3}x - 4$
$\frac{1}{2}x + y = 4$	$\frac{x}{8} + \frac{y}{4} = 1$	$\mathrm{y} = \dfrac{-1}{2}\mathrm{x} + 4$



Hint: The intercept form is the easiest to plot for the first two lines.

The next part is a short derivation of the point form given the parametric form of a line. Write out your steps. Start with the parametric form of a line. Solve for t in both coordinates. Then set the two expressions to be equal to obtain the point form.

<u>**Solution:**</u> Given the parametric form

$$x = x_0 + t \cdot (x_1 - x_0), \text{ and}$$

$$y = y_0 + t \cdot (y_1 - y_0),$$

solving for t in both gives

$$t = \frac{x - x_0}{x_1 - x_0}, \text{ and} t = \frac{y - y_0}{y_1 - y_0}.$$

A given point (x, y) is generated from a single value of t, so we set these two equations for t equal to see that

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0},$$

the point form of a line.

3 Quadratic Interpolation

Find a quadratic function $y = ax^2 + bx + c$ passing through the points $(-1, 4)$, $(3, 8)$, and $(5, 7)$.

To find the function, substitute the points' x values to obtain a system of three linear equations in a, b, and c. Then solve the linear system, preferably by the elimination method. Finally, write the function $y = ax^2 + bx + c$ given your solution for a, b, and c.

Show all your work. The grade depends more on your work than on the actual solution.

Substituting the points provides the system of equations

$$4 = a - b + c,$$

$$8 = 9a + 3b + c, \text{ and}$$

$$7 = 25a + 5b + c.$$

Translating into an augmented matrix:

$$\begin{bmatrix} 1 & -1 & 1 & 4 \\ 9 & 3 & 1 & 8 \\ 25 & 5 & 1 & 7 \end{bmatrix}$$

Start eliminating from the last column; it's slightly easier. Subtracting the first row from each of the second and third rows:

$$\begin{bmatrix} 1 & -1 & 1 & 4 \\ 8 & 4 & 0 & 4 \\ 24 & 6 & 0 & 3 \end{bmatrix}$$

Now multiply the second row by 1/4 and the last by 1/3:

$$\begin{bmatrix} 1 & -1 & 1 & 4 \\ 2 & 1 & 0 & 1 \\ 8 & 2 & 0 & 1 \end{bmatrix}$$

Subtracting twice the second row from the last:

$$\begin{bmatrix} 1 & -1 & 1 & 4 \\ 2 & 1 & 0 & 1 \\ 4 & 0 & 0 & -1 \end{bmatrix}$$

Now dividing the last by 4:

$$\begin{bmatrix} 1 & -1 & 1 & 4 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1/4 \end{bmatrix}$$

Eliminating upwards:

$$\begin{bmatrix} 0 & 0 & 1 & 23/4 \\ 0 & 1 & 0 & 3/2 \\ 1 & 0 & 0 & -1/4 \end{bmatrix}$$

So the final solution is $\mathbf{a = -1/4 = -0.25}$, $\mathbf{b = 3/2 = 1.5}$, and $\mathbf{c = 23/4 = 5.75}$. So $\mathbf{y = -0.75x^2 + 1.5x + 5.75}$. You can (and should) check this by substituting the points.

4 Slopes: Parallel, Perpendicular, *etc.*

Each row of the following table provides a line in standard form and a point. Given the following lines in standard form and a point for each line, find the equation of a line parallel to the given one through the point and the equation of a line perpendicular to the given line through the point. **All results must be in standard form.**

Line	Point	Parallel	Perpendicular
$3x - 4y = 8$	$(8, 11)$	$3x - 4y = -20$	$4x + 3y = 65$
$8x + 7y = -5$	$(-2, 7)$	$8x + 7y = 33$	$7x - 8y = -70$
$\frac{3}{5}x - \frac{6}{5}y = 2$	$(0, 0)$	$\frac{3}{5}x - \frac{6}{5}y = 0$	$\frac{6}{5}x + \frac{3}{5}y = 0$
(alternate)		$x - 2y = 0$	$2x + y = 0$
$\frac{6}{7}x + \frac{1}{7}y = -3$	$(-1, -1)$	$\frac{6}{7}x + \frac{1}{7}y = -1$	$\frac{1}{7}x - \frac{6}{7}y = \frac{5}{7}$
(alternate)		$6x + y = -7$	$x - 6y = 5$

Describe the pattern you see in the coefficients.

Answer: The coefficients for the parallel form are identical (or multiplied by a constant), and the coefficients for the perpendicular form are swapped *(and possibly multiplied by a constant)* with one coefficient negated.

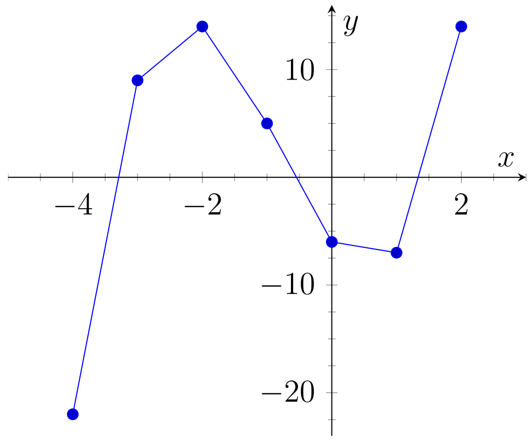
Answer the following:

- What is the slope of a line connecting $(-1, -1)$ and $(2, 2)$? **The slope is 1.**
- What is the slope of a line connecting $(-1, 1)$ and $(2, -2)$? **The slope is -1.**
- What is the slope of a horizontal line? **The slope is 0.**
- What is the slope of a vertical line? **The slope is undefined.**
- If two lines have different slopes, how often do they intersect? **Exactly once.**
- If two lines have the same slope, what are the two cases describing their possible intersections? **Either never or they are the same line.**

5 Lines and Roots of Polynomials

Consider the polynomial $f(x) = 2x^3 + 5x^2 - 8x - 6$. Fill in the following table by evaluating the polynomial at the provided points. Then plot the line segments between the points $(x, f(x))$.

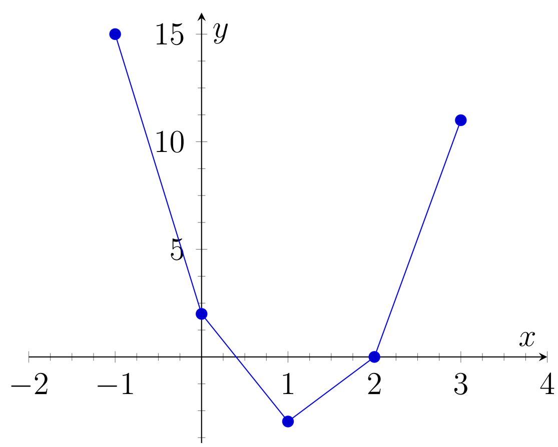
x	$f(x)$
-4	-22
-3	9
-2	14
-1	5
0	-6
1	-7
2	14



In which intervals are there roots of f(x)? In other words, in which intervals does $f(x) = 0$ somewhere within the interval? You do not need to determine the roots here, only the intervals. **Answer: The intervals are $(-4, -3)$, $(-1, 0)$, and $(1, 2)$.**

Now consider $g(x) = 4x^2 - 9x + 2$. Evaluate the function at the following points and plot the segments.

x	$g(x)$
-1	15
0	2
1	-3
2	0
3	11



Now find the non-obvious root by bisection. One root, where $g(x) = 0$, will occur at one of the x values in the above table. Take the interval that must contain the other root. Evaluate the function at the half-way point and decide on which side the root must lie. Continue until you find the root where $g(x) = 0$. Verify the roots you find by comparing to those from the quadratic equation,

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Answer: $g(0.5) = -1.5$, $g(0.25) = 0$, so the other root is $\mathbf{1/4}$. This and the obvious root of 2 agree with the quadratic equation.

6 Taxi Fares: A Linear Model

A summary of taxi fares in four major cities:

city	initial charge	initial distance	distance increment	cost per incr.
San Francisco	2.85	1/5	1/5	0.45
New York	2.50	1/5	1/5	0.40
Los Angeles	2.20	1/11	1/11	0.20
Las Vegas	3.20	1/8	1/8	0.25

So for a taxi ride in New York, you pay \$2.50 for the first $1/5^{\text{th}}$ of a mile and \$0.45 per each additional $1/5^{\text{th}}$ of a mile.

Express these taxi fares as linear models. That is, for each city, express the total fare as a line where the y coordinate is the total charge and the x coordinate is the total distance. The line will pass through the initial point (initial distance, initial charge) and will increase by an appropriate increment per mile. *Ignore the fact that the line dips below the minimum fare for points before the initial distance.*

The line will have the form $F = mD + b$, where F is the fare in dollars and D is the distance in miles. You must find m and b.

city	$F = mD + b$
San Francisco	**F = 2.25 D + 2.4**
New York	**F = 2 D + 2.1**
Los Angeles	**F = 2.2 D + 2**
Las Vegas	**F = 2 D + 2.95**

Each of these was obtained from the point-slope form, $\mathbf{y - y_0 = m(x - x_0)}$, where the initial point is the initial distance and initial charge and the slope is the cost per increment over the distance increment.

Use the models to answer the following questions:

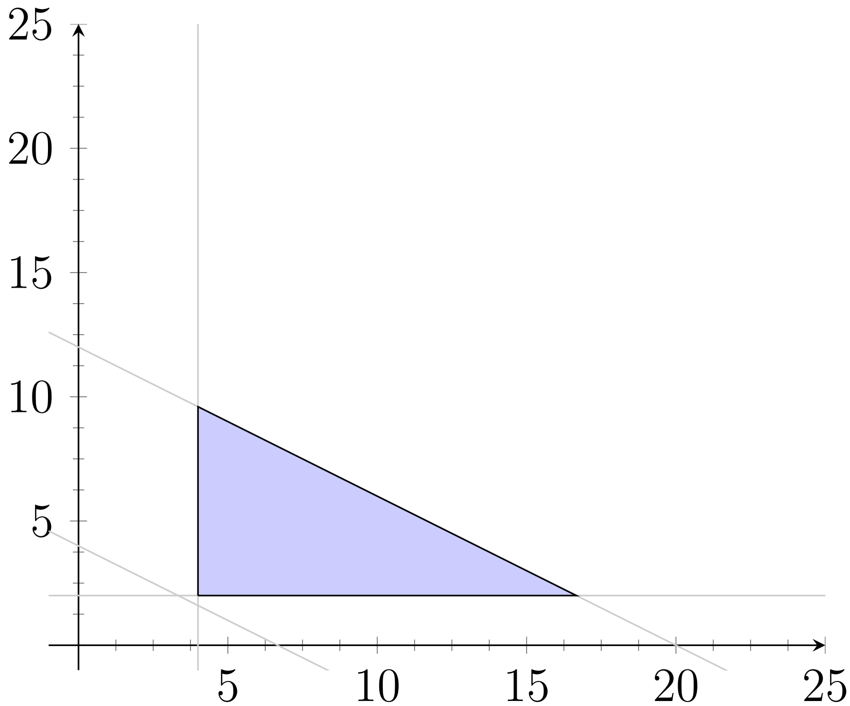
1. How many miles must you travel for the fares to be equal in San Francisco and Las Vegas? What is the distance and fare where they are equal? **2.2 miles, \$7.35**
2. How many miles must you travel for the fares to be equal in New York and Los Angeles? What is the distance and fare where they are equal? **.5 miles, \$3.10**
3. Are taxis more expensive in San Francisco or Los Angeles? *Hint: Plot the lines and see if one is always above the other.* **San Francisco**

7 Linear Inequalities

Say you are ordering lights. You need at least four low-intensity lights and two high-intensity lights. A low-intensity light costs \$15, and a high-intensity light costs \$25. The company from which you purchase the lights requires a minimum order of \$100. You have a total budget of \$300.

Let x be the number of low-intensity lights, and let y be the number of high-intensity lights. Write the problem above as a set of linear inequalities in x and y. Then graph the feasible region.

Answers: $x \geq 4$, $y \geq 2$, $15x + 25y \geq 100$, $15x + 25y \leq 300$. The requirement that $15x + 25y \geq 100$ is redundant because $15 \cdot 4 + 25 \cdot 2 = 110 \geq 100$.



Low-intensity lights give off 1000 lumen, and high intensity lights give off 2000 lumen. **So the objective function is $\mathbf{1000x + 2000y}$**.

1. What is the maximum intensity of lumen you can buy? Provide not only the lumen but also a point in the graph above that achieves that quantity.

2. What is the minimum intensity of lumen you can buy? Provide not only the lumen but also a point in the graph above that achieves that quantity.

Answer: There are three interesting points, $(2, 2)$, $(50/3, 2)$, and $(4, 48/5)$. At these points, the objective is 6000, $20666.\overline{6}$, and 24200, respectively. So the most lumen is **24200** at $(\mathbf{4}, \mathbf{48/5})$, and the least lumen is **8000** at $(\mathbf{4}, \mathbf{2})$.

8 Algebraic Transformations

Justify each line in the following derivation of $1 = 2$, or state the mistake. One of the lines already is justified (and is correct). Also continue the column of numerical examples; that column should help you identify the mistake[1].

	Reason	Example
$x = y$	Given	$3 = 3$
$x^2 = xy$	<u>2</u>	$3^2 = 3 \cdot 3$
$x^2 - y^2 = xy - y^2$	<u>3</u>	$\underline{3^2 - 3^2 = 3 \cdot 3 - 3^2}$
$(x + y)(x - y) = y(x - y)$	Factoring.	$\underline{(3 + 3) \cdot (3 - 3) = 3 \cdot (3 - 3)}$
$x + y = y$	<u>6</u>	$\underline{3 + 3 = 3}$
$y + y = y$	<u>1</u>	$\underline{3 + 3 = 3}$
$2y = y$	<u>5</u>	$\underline{2 \cdot 3 = 3}$
$2 = 1.$	<u>4</u> or <u>6</u>	$\underline{2 = 1}$

1. Substituting using the given information.
2. Multiplication by the same, non-zero quantity.
3. Addition or subtraction of the same quantity.
4. Division by the same, non-zero quantity.
5. Simplifying one or both sides.
6. The mistake.

Explain why the mistake is a mistake.

<u>Answer:</u> The quantity $x - y = 0$, as can be seen through the numerical example $3 - 3 = 0$. **And if you answered that the last line was also a mistake:** The problem never stated that $y \neq 0$, so this also could be division by zero.

[1]**<u>Note:</u>** Because I never stated $y \neq 0$, the last line also could be a mistake. That wasn't intended, but I accept that answer.