Solutions for the fourth week's homework
Math 202

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Also available as PDF.

Note: These are my approaches to these problems. There are many ways to tackle each.

1 Problem set 2.2

1.1 Problem 1

(a) June 13$^{\text{th}}$: nominal, first: ordinal

(b) eleventh: ordinal, second: ordinal, 6-iron: nominal, 160 yards: cardinal

(c) 7 pin: nominal, sixth frame: ordinal, 9: cardinal

1.2 Problem 2

(a) There are five letters in the phrase, {P,A,N,M,B}, so we can associate $1 \leftrightarrow$ P, $2 \leftrightarrow$ A, *etc.* The two sets are **equivalent**.

(b) The second set has one more element than the first, so the sets are **not equivalent**.

(c) We can associate $o \leftrightarrow t$, $n \leftrightarrow w$, and $e \leftrightarrow o$, where the left quantities come from $\{o, n, e\}$ and the right from $\{t, w, o\}$. The two sets are **equivalent**.

(d) $\{0\}$ has in element, and $\emptyset$ has none, so the sets are **not equivalent**.

1.3 Problem 6

(a) We can use the function $f(w) = w + 1$ to construct a bijection between W and N.

(b) The function $f(d) = d + 1$ creates a one-to-one mapping between D and E.

(c) The function $f(n) = 10^n$ creates a one-to-one mapping between N and $\{10, 100, \ldots\}$.

1.4 Problem 13

The drawing will have

- $|B \cap C| - |A \cap B \cap C| = 5$ in the unlabeled portion of $B \cap C$,
- $|B| - |A \cap B| - |B \cap C| + |A \cap B \cap C| = 28$ in the remaining unlabeled portion of B,
- $|A \cap C| - |A \cap B \cap C| = 8$ in the unlabeled portion of $A \cap C$,
- $|A| - |A \cap B| - |A \cap C| + |A \cap B \cap C| = 15$ in the remaining unlabeled portion of A,
- $|C| - |B \cap C| - |A \cap C| + |A \cap B \cap C| = 10$ in the remaining unlabeled portion of C, and
- $|U| - (10 + 7 + 5 + 28 + 8 + 15 + 10) = 17$ in U but outside A, B, and C.

1.5 Problem 21

(a) The function $f(a) = a$ is the one-to-one correspondence from A to itself.

(b) If $A \sim B$, there is a function $f(a) = b$ where each $b \in B$ appears for exactly one $a \in A$ and the function is defined for every $b \in B$ and $a \in A$. The *inverse* function $g(b) = a$ where $b = f(a)$ shows $B \sim A$.

(c) If $A \sim B$ and $B \sim C$, then there are one-to-one functions $f(a) = b$ and $g(b) = c$ for each mapping. The function $h(a) = g(f(a))$ then is a one-to-one mapping between A and C.

1.6 Problem 23

For **(a)** and **(b)**, the table is Pascal's triangle, which we already have in the notes. For **(c)**, we want the entry $P_{6,3}$ in our earlier notation. There are 20 such ways.

1.7 Why answering problem 32 would be a bad idea.

Wow. Never, **ever** give someone a list of all your identifying numbers. Identity theft is a serious problem. While quite often all these numbers are available for a little work, at least make a criminal work for them.

2 Problem set 2.3

2.1 Problem 2

Substituting into $|A \cup B| = |A| + |B| - |A \cap B|$, we see that $10 = 5 + 8 - |A \cap B|$, or that $|A \cap B| = 3$.

2.2 Problem 5

I'm just going to show these as lines. Illustrating them on a "number line" is equivalent. A better diagram for the last would wrap portions of the result in parentheses to show how the line extended.

(a) $3 + 5 =$ ●—●—●—● + ●—●—●—●—●—●
= ●—●—●—●—●—●—●—●—●
$= 8$

(b) $5 + 3 =$
●—●—●—●—●—● + ●—●—●—●
= ●—●—●—●—●—●—●—●—●
$= 8$

(c) $4 + 2 =$
●—●—●—●—● + ●—●—●
= ●—●—●—●—●—●—●
$= 6$

(d) $0 + 6 =$
● + ●—●—●—●—●—●—●
= ●—●—●—●—●—●—●
$= 6$

(e) $3 + (5 + 7) =$
●—●—●—● + (●—●—●—●—●—● + ●—●—●—●—●—●—●—●)
= ●—●—●—● + ●—●—●—●—●—●—●—●—●—●—●—●—●
= ●—●—●—●—●—●—●—●—●—●—●—●—●—●—●—●
$= 15$

(f) $(3 + 5) + 7 =$
(●—●—●—● + ●—●—●—●—●—●) + ●—●—●—●—●—●—●—●
= ●—●—●—●—●—●—●—●—● + ●—●—●—●—●—●—●—●
= ●—●—●—●—●—●—●—●—●—●—●—●—●—●—●—●
$= 15$

2.3 Problem 11

Again, I'm just going to show these as lines. Illustrating them on a "number line" is equivalent.

(a) $7 - 3 =$
•—•—•—•—•—•—•—• – •—•—•—•
= •—•—•—•—•
$= 4$

(b) $7 - 4 =$
•—•—•—•—•—•—•—• – •—•—•—•—•
= •—•—•—•
$= 3$

(c) $7 - 7 =$
•—•—•—•—•—•—•—• – •—•—•—•—•—•—•—•
= • $= 0$

(d) $7 - 0 =$
•—•—•—•—•—•—•—• - •
= •—•—•—•—•—•—•—•
$= 7$

2.4 Problem 24

Draw with two shadings and show that the intersection is shaded twice.

3 Write 2 + 3 using disjoint sets.

If we let $2 \equiv \{a, b\}$ and $3 \equiv \{c, d, e\}$, then $2+3 \equiv \{a, b\} \cup \{c, d, e\} = \{a, b, c, d, e\}$.

4 Illustrate 2 + 3 using Peano arithmetic.

We defined addition with

$$\begin{aligned} a + 0 &= a, \text{ and} \\ a + S(b) &= S(a + b). \end{aligned}$$

Here, $3 = S(2)$, so

$$\begin{aligned}
2 + 3 &= 2 + S(2) \\
&= S(2 + 2) \\
&= S(2 + S(1)) \\
&= S(S(2 + 1)) \\
&= S(S(2 + S(0))) \\
&= S(S(S(2 + 0))) \\
&= S(S(S(2))) \\
&= 5.
\end{aligned}$$

5 Problem set 2.4

5.1 Problem 5

(a) Any set that contains only 1 and some other number is **closed** because 1 is the multiplicative identity.

(b) Any set that contains only 1 and some other number is **closed** because 1 is the multiplicative identity.

(c) $2 \cdot 4 = 8 \notin \{0, 2, 4\}$, so this is **not closed**.

(d) The product of even numbers always is even, so this is **closed**.

(e) The product of odd numbers cannot be even, so they must be odd and this set is **closed**.

(f) $2^2 \cdot 2^3 = 2^5 \notin \{1, 2, 2^2, 2^3\}$, so this set is **not closed**.

(g) The product of powers of two is a power of two, so this set is **closed**.

(h) Similarly, the product of powers of seven is a power of seven so this set is **closed**. Both of these are closed because $\{0, 1, 2, \ldots\}$ is closed under addition; $a^i \cdot a^j = a^{i+j}$, moving the property up to the superscript.

5.2 Problem 10

Each product is equal to the appropriate shaded area, and the sum is equal to the entire rectangle. The letters correspond directly to the positions.

I had completely forgotten about the FOIL mnemonic. After long enough, it's just a part of what you do.

5.3 Problem 26

The operation is **closed** because no new shapes are introduced in the operator's table of results.

The operation is **commutative** because the operation table is symmetric across the diagonal axis. Thus $a \star b = b \star a$.

Because $\circ \star x = x$, the $\circ$ symbol is $\star$'s identity.

After the identity, there are only two symbols remaining. Thus the commutative property combined with the identity means that *this* operator must be **associative**.

6 Illustrate $2 \cdot 3$ using Peano arithmetic. You do not need to expand addition.

We defined multiplication with

$$\begin{aligned} a \cdot 0 &= 0, \text{ and} \\ a \cdot S(b) &= a + (a \cdot b). \end{aligned}$$

So

$$\begin{aligned} 2 \cdot 3 &= 2 \cdot S(2) \\ &= 2 + (2 \cdot 2) \\ &= 2 + (2 \cdot S(1)) \\ &= 2 + (2 + (2 \cdot 1)) \\ &= 2 + (2 + (2 \cdot S(0))) \\ &= 2 + (2 + (2 + 2 \cdot 0)) \\ &= 2 + 2 + 2 = 6. \end{aligned}$$

7 Illustrate $(1 \cdot 2) \cdot 3 = 1 \cdot (2 \cdot 3)$ using a volume of size six.

Draw two $1 \times 2 \times 3$ boxes made of $1 \times 1 \times 1$ boxes. To illustrate $(1 \cdot 2) \cdot 3$, show that the 1×2 portion is stacked 3 times. To illustrate $1 \cdot (2 \cdot 3)$, show that the 2×3 portion is stacked across 1 time.