Solutions for the seventh week's homework
Math 202

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Also available as PDF.

Note: These are my approaches to these problems. There are many ways to tackle each.

1 Problem set 4.1

Problem 2 The (non-repeated) factorizations are $1 \cdot 35$ and $5 \cdot 7$. Drawing those as boxes is straight-forward.

Problem 4

Factors of 18	1	2	3	6	9	18
Quotient	18	9	6	3	2	1

Problem 7

- $48 = 1 \cdot 48 = 2 \cdot 24 = 3 \cdot 16 = 4 \cdot 6$, so the factors are 1, 2, 3, 4, **6**, 16, 24, and 48.
- $54 = 1 \cdot 54 = 2 \cdot 27 = 3 \cdot 18 = 6 \cdot 9$, so the factors are 1, 2, 4, **6**, 9 18, 27, 54.
- The largest common factor then is **6**.

Problem 9

- $48 = 2^4 \cdot 3^1$
- $108 = 2^2 \cdot 3^3$
- $2250 = 2^2 \cdot 3^2 \cdot 5^3$
- $24\,750 = 2^1 \cdot 3^2 \cdot 5^3 \cdot 11^1$

Problem 10

- Yes, because all primes are shared and no powers of the primes exceed those of a.
- No, because 3^2 has a larger exponent than the 3^1 in a.
- $a/b = (2^3 \cdot 3^1 \cdot 7^2)/(2^2 \cdot 3^1) = 2^{3-2} \cdot 3^{1-1} \cdot 7^{2-0} = 2^1 \cdot 7^2$.
- There are $(2+1) \cdot (1+1) \cdot (2+1) = 18$ factors of a.
- We can make a list by running up the exponents:

$$
\begin{gathered}
2^0 \cdot 3^0 \cdot 7^0 \\
2^1 \cdot 3^0 \cdot 7^0 \\
2^2 \cdot 3^0 \cdot 7^0 \\
2^3 \cdot 3^0 \cdot 7^0 \\
2^0 \cdot 3^1 \cdot 7^0 \\
2^1 \cdot 3^1 \cdot 7^0 \\
2^2 \cdot 3^1 \cdot 7^0 \\
2^3 \cdot 3^1 \cdot 7^0 \\
2^0 \cdot 3^0 \cdot 7^1 \\
2^1 \cdot 3^0 \cdot 7^1 \\
2^2 \cdot 3^0 \cdot 7^1 \\
2^3 \cdot 3^0 \cdot 7^1 \\
2^0 \cdot 3^1 \cdot 7^1 \\
2^1 \cdot 3^1 \cdot 7^1 \\
2^2 \cdot 3^1 \cdot 7^1 \\
2^3 \cdot 3^1 \cdot 7^1 \\
2^0 \cdot 3^0 \cdot 7^2 \\
2^1 \cdot 3^0 \cdot 7^2 \\
2^2 \cdot 3^0 \cdot 7^2 \\
2^3 \cdot 3^0 \cdot 7^2 \\
2^0 \cdot 3^1 \cdot 7^2 \\
2^1 \cdot 3^1 \cdot 7^2 \\
2^2 \cdot 3^1 \cdot 7^2 \\
2^3 \cdot 3^1 \cdot 7^2
\end{gathered}
$$

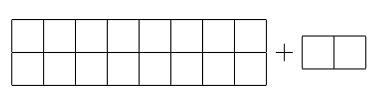
Problem 13 No, it is only true that at least **one** prime factor cannot exceed $\sqrt{n}$. Consider $2 \cdot 47 = 94$, where both 2 and 47 are prime. We have $\sqrt{94} < 10$ but $47 > 10 > \sqrt{94}$. But $2 < \sqrt{94}$.

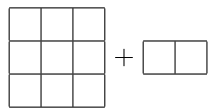
Problem 14 No. If n is not prime, some of its factors may split between b and c. For example, $2 \cdot 3 = 6 \mid 18 = 2 \cdot 9$, but $6 \nmid 2$ and $6 \nmid 9$. The factors of $n = 6$, 2 and 3, are split between $a = 2$ and $b = 9$.

Problem 23 Here it **is** true. Consider the prime factorizations of b and c. If $p \mid bc$, then p must appear in one or both of those prime factorizations, and thus it must divide at least one of b and c.

Problem 24 Again, use the prime factorization of n. Because p and q are primes, they must appear in that factorization. Then $pq \mid n$ because both appear, so you can commute products around to group (pq) and the rest of the factorization.

2 Two diagrams

$8 \nmid 18$**:** $18 = 2 \cdot 8 + 2$, so you can draw and count: 

$3 \nmid 11$: $11 = 3 \cdot 3 + 2$: 

3 Problem set 4.2

Problem 1

- 1554 is even, so **divisible by 2**, does not end in 5 or 0, so **is *not* divisible by 5**, and has digits that add to 0 (mod 3), so **is divisible by 3**.
- 1999 is not even, so **is *not* divisible by 2**, does not end in 5 or 0, so **is *not* divisible by 5**, and has digits that add to 1 (mod 3), so **is *not* divisible by 3**.
- 805 **is *not* divisible by 2**, **is divisible by 5**, and **is *not* divisible by 3**.
- 2450 **is divisible by 2**, **is divisible by 5**, and **is *not* divisible by 3**.

Problem 2

- 2 and 3
- 2 and 5
- 3 and 5
- 2, 3, and 5

Problem 8 The missing digit must be divisible by 2. Also, the sum of the digits must be congruent to 0 modulo 3. The non-blank digits already add to 0 (mod 3), so the missing digit must be a multiple of 3. There is only one even multiple of 3 less than 10, so the missing digit must be **6**.

Problem 14 Expanding the positional notation and simplifying, $abc, abc = a \cdot (10^5 + 10^2) + b \cdot (10^4 + 10^1) + c \cdot (10^3 + 10^0) = (a \cdot 10^2 + b \cdot 10^1 + c) \cdot (10^3 + 10^0) = abc \cdot 1001$. Now $1001 = 7 \cdot 11 \cdot 13$, so abc, abc is divisible by each of those.

Problem 15

- $ab - ba = a \cdot 10 + b - (b \cdot 10 + a) = (a - b) \cdot 10 + (b - a)$. Because $10 \equiv 1 \pmod 9$, this becomes $a - b + b - a \equiv 0 \pmod 9$. The result always is a multiple of 9. Equivalently, we can rearrange $(a - b) \cdot 10 + (b - a) = (a - b) \cdot 10 + -1 \cdot (a - b) = (a - b) \cdot (10 - 1) = 9(a - b)$, giving also *which* multiple of 9.
- Here the difference is $(a - c) \cdot 10^2 + 0 + (c - a)$ because the middle digit always cancels. Again, the result always is a multiple of 9 and we can rearrange $(a - c) \cdot 10^2 + 0 + (c - a) = (a - c) \cdot 10^2 + -1 \cdot (a - c) = (a - c) \cdot (10^2 - 1) = 99(a - c)$ to see the result is a multiple of 99.

4 A familiar incomplete integer

> Take a familiar incomplete integer, _679_. Using the expression of _679_ as $N = 10^4 \cdot x_4 + x_0 + 6790$, use $8 \mid N$ to find x_0. Given that, use $9 \mid N$ to find x_4. Now if 72 turkeys cost $\$$_679_, what is the total?

If $8 \mid N$ then 8 divides the last three digits, so $8 \mid 790 + x_0$. Thus $790 + x_0 \equiv 0 \pmod 8$. Because $790 \equiv 6 \pmod 8$, we know that $x_0 \equiv 2 \pmod 8$. The only decimal digit satisfying $x_0 \equiv 2 \pmod 8$ is $\mathbf{x_0} = \mathbf{2}$.

Now we have $N = 10^4 \cdot x_4 + 6792$. For $9 \mid N$, the sum of the digits must be zero modulo 9. Thus $x_4 + 6 + 7 + 9 + 2 \equiv 0 \pmod 9$, or $x_4 + 6 \equiv 0 \pmod 9$. Thus $\mathbf{x_4} = \mathbf{3}$, and $\mathbf{N} = \mathbf{36792}$.

(I forgot the decimal place in the problem, so these are very expensive turkeys.)

So if 72 turkeys cost $36792, each turkey costs $511. If I had remembered the decimal place correctly, the turkeys cost $5.11 each.