# Math 202 Final Exam

6 December, 2008

Following are twelve questions, each worth the same amount. Complete ***five*** of your choice. I will only grade the first ***five*** I see. Make sure your name is on the top of each page you return.

Explain your reasoning for each problem whenever appropriate; that helps me give partial credit. Perform scratch work on scratch paper; keep your explanations clean.

Make final answers obvious by boxing or circling them. When a question asks you to construct a table or perform a computation, showing the table or writing out the computation's steps is a part of the question and is **not optional**.

And remember to read and answer the entire question. There is copious explanation before a few problems. The explanation repeats some relevant material from class.

## Contents

# 1 Problem solving

*This problem is based on one from G. Pólya's "How to Solve It".*

Among your papers, a grandchild finds a bill:

> Four free-range, heritage turkeys, 15 pounds each, for \$__.______ per pound: \$__62.2__.

The first and last digits of the number that obviously represented the total price of those fowls are replaced here by blanks, for they have faded and are now illegible.

What are the two faded digits in the total, and what was the price of one turkey? Explain your reasoning.

*Hint: The total is less than \$500. The price per turkey is not a whole number of dollars, but it is a whole number of cents. There are many ways to solve this; you only need one.*

Structure your explanation according to Pólya's principles: **understanding the problem**, **devising a plan**, **carrying out the plan**, and **looking back**. These need not be in order, but each observation or method should be attributed to one phase.

## 2 Set theory: operations and relations

Define the result of the following operations using set-builder notation, and *also provide an English description of when an element is in the result*:

- $A \cap B$
- $A \setminus B$
- $(A \setminus B) \cap C$

Draw two Venn diagrams for each of the following illustrating different ways $A$, $B$, and $C$ can be related (*e.g.* one is a subset of another, all sets are distinct, or other possibilities):

- $A \cup B$
- $A \setminus B$
- $(A \setminus C) \cup (B \setminus C)$

## 3 Set theory: definitions and relations

Answer clearly:

- Write out the set $\{3x-2 \,|\, x \text{ is an integer}, -1 < x \leq 3\}$ by listing all its elements appropriately.
- Write the following set succinctly in set-builder notation: $\{4, 7, 10, 13, 16, \ldots\}$.
- Given two sets $A$ and $B$, when is $A \subset B$?
- Given two sets $A$ and $B$, when is $A \supset B$?
- What is a *proper* subset?
- What are two ways to write an empty set symbolically?

Fill in each with the most appropriate relation ($\in$, $\subset$, $=$, or no relation at all):

- $\emptyset$ __ $\{x \,|\, x \text{ is an odd number divisible by two}\}$
- $1$ __ $\{x/4 \,|\, x \text{ is a whole number less than five}\}$
- $\emptyset$ __ $A$ for all sets $A$
- $\{6\}$ __ $\{1, 2, 3, 4, 5, 6\}$
- $6$ __ $\{1, 2, 3, 4, 5, 6\}$

Under what circumstances are each of the following true for sets $A$ and $B$?

- $A \cap B = A$
- $A \subset B$ for any set $B$
- $A \setminus B = B$

# 4 Properties of Operations

Is addition **closed** over odd numbers? If not, provide an example and state why addition is not closed over odd numbers.

For each of the following properties of operations, provide one example using numbers and one using set theory. The examples can be either symbolic or concrete.

| Property | Numbers | Set theory |
|---|---|---|
| **commutative** | ____________ | ____________ |
| **associative** | ____________ | ____________ |
| **distributive** | ____________ | ____________ |

For each of the following operations, provide the associated identity element:

| Operation | Identity |
|---|---|
| **numerical addition** | ____________ |
| **numerical multiplication** | ____________ |
| **set union** | ____________ |

# 5 Repeating Decimals

Another way to find the period of a decimal expansion is to consider powers of ten. To find the period of a fraction $\frac{1}{d}$, examine powers of ten modulo $d$. For example, consider $\frac{1}{3} = 0.\overline{3}$ and $\frac{1}{7} = 0.\overline{142857}$. The period of $0.\overline{3}$ is one, and the period of $0.\overline{142857}$ is six.

The following tables show the powers of ten modulo three and seven, respectively:

| $i$ | $10^i \equiv \ldots \pmod 3$ |
|---|---|
| 0 | $1 \equiv 1 \pmod 3$ |
| 1 | $10 \equiv 1 \pmod 3$ |
| | $\vdots$ |

| $i$ | $10^i \equiv \ldots \pmod 7$ |
|---|---|
| 0 | $1 \equiv 1 \pmod 7$ |
| 1 | $10 \equiv 3 \pmod 7$ |
| 2 | $100 \equiv 2 \pmod 7$ |
| 3 | $1000 \equiv 6 \pmod 7$ |
| 4 | $10000 \equiv 4 \pmod 7$ |
| 5 | $100000 \equiv 5 \pmod 7$ |
| 6 | $1000000 \equiv 1 \pmod 7$ |
| | $\vdots$ |

In each table, a number eventually appears twice. The distance between those two appearances is equal to the decimal expansion's period. For 3 that difference is 1 (from 0 to 1), and for 7 the difference is 6 (from 0 to 6). Reasoning *inductively*, assume this is true. Note that the numbers in the table are *not* the same as the digits in the expansion.

- Construct similar tables to find the periods of $\frac{1}{11}$ and $\frac{1}{21}$. (You can check if you found the correct period by computing 1/11 and 1/21.)
- Remembering that $\frac{1}{15} = 0.0\overline{6}$, construct a table showing that the period of $\frac{1}{15}$ is 1. What number repeats in the table? What can you say about the number of digits *before* the expansion $\frac{1}{15} = 0.0\overline{6}$ repeats?

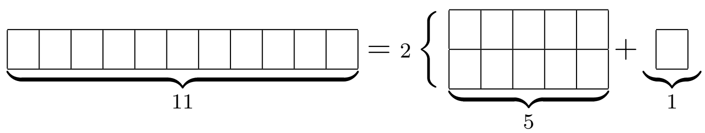
# 6 Divisibility

The division form or division algorithm expresses one integer $b$ in terms of another $a \neq 0$,

$$b = q \cdot a + r \quad \text{where} \quad 0 \leq r < |a|.$$

The quantity $q$ is the *quotient* and $r$ is the *remainder*. We say that $a$ divides $b$, or $a \mid b$ if $r = 0$ in this form.

We can illustrate the division form with boxes. For example, we can draw $11 = 2 \cdot 5 + 1$ as




Complete the following division forms, answer, illustrate each with a block diagram:

- $18 =$ ______ $\cdot\, 5 +$ ______. Does $5 \mid 18$?
- $18 =$ ______ $\cdot\, 6 +$ ______. Does $6 \mid 18$?
- $18 =$ ______ $\cdot\, 7 +$ ______. Does $7 \mid 18$?

We can use modular arithmetic (arithmetic with remainders) and positional notation ($123 = 1 \cdot 10^2 + 2 \cdot 10^1 + 3 \cdot 10^0$) to assist with some easier divisibility rules.

Because $10 \equiv 1 \pmod 9$, we can expand $123 = 1 \cdot 10^2 + 2 \cdot 10^1 + 3 \cdot 10^0 \equiv 1 \cdot 1^2 + 2 \cdot 1^1 + 3 \cdot 1^0 \equiv 1 + 2 + 3 \equiv 6 \pmod 9$, so $3 \nmid 123$.

- Using a similar method, does 3 divide 382030278639098?
- Does 9 divide $382\underline{30}02\underline{8}86390\underline{89}$? (The underlined digits are different.)
- State an easy rule in English for divisibility by 3.
- What is $10^k \pmod 2$ for $k = 1$ and for integers $k > 1$? So what is a divisibility rule for 2?

# 7 Rational Arithmetic

The method for adding fractions can be derived as follows:

$$\begin{aligned} \frac{a}{b}+\frac{c}{d} &= \frac{a}{b}\cdot 1+\frac{c}{d}\cdot 1 && \underline{\qquad} \\ &= \frac{a}{b}\cdot\frac{d}{d}+\frac{c}{d}\cdot\frac{b}{b} && \underline{\qquad} \\ &= \frac{ad}{bd}+\frac{cb}{db} && \underline{\qquad} \\ &= \frac{ad}{bd}+\frac{cb}{\mathbf{bd}} && \underline{\qquad} \\ &= \frac{ad+cb}{bd}. && \underline{\qquad} \end{aligned}$$

(The bold is to emphasize what changed in one line above.)

Justify each line in the derivation by labeling it with one of the following arithmetic properties or rules:

1. additive identity
2. multiplicative identity
3. commutativity of addition
4. commutativity of multiplication
5. distributivity of multiplication over addition
6. multiplying fractions
7. $1 = \frac{d}{d}$ for any non-zero $d$
8. adding fractions of equal denominators

Compute and reduce to lowest terms by removing common factors from the numerator and denominator:

$$\frac{1}{2}+\frac{1}{3} = \underline{\qquad} \qquad \frac{9}{14}\cdot 7 = \underline{\qquad}$$

$$\frac{1}{2}-\frac{1}{3} = \underline{\qquad} \qquad \frac{9}{14}\cdot\frac{7}{3} = \underline{\qquad}$$

$$\frac{7}{6}-\frac{2}{3} = \underline{\qquad} \qquad \frac{9}{14}\Big/\frac{7}{3} = \frac{9}{14}\div\frac{7}{3} = \underline{\qquad}$$

# 8 Prime factorization, the GCD, and the LCM

Factor the following numbers into products of primes raised to powers:

$$\begin{aligned} 72 &= 2^3 \cdot 3^2 \\ 36 &= 2^2 \cdot 3^3 \\ 154 &= \underline{\qquad} \\ 442 &= \underline{\qquad} \\ 476 &= \underline{\qquad} \\ 4235 &= \underline{\qquad} \end{aligned}$$

Recall that the greatest common divisor of $a$ and $b$, written $(a, b)$, is the largest integer that divides both $a$ and $b$. The least common multiple, $\mathrm{lcm}(a, b)$, is the least *positive* integer divisible by $a$ and $b$.

Using the factorizations above, provide the following greatest common divisors in factored and numerical forms:

$$\begin{aligned} (72, 30) &= 2 \cdot 3 && = 8 \\ (72, 36) &= \underline{\qquad} && = \underline{\qquad} \\ (442, 476) &= \underline{\qquad} && = \underline{\qquad} \\ (154, 476) &= \underline{\qquad} && = \underline{\qquad} \\ (476, 4235) &= \underline{\qquad} && = \underline{\qquad} \\ (154, 476, 4235) &= \underline{\qquad} && = \underline{\qquad} \end{aligned}$$

Also using the factorizations above, provide the following least common multiples in *factored* form only:

$$\begin{aligned} \mathrm{lcm}(72, 36) &= 2^2 \cdot 3^2 \\ \mathrm{lcm}(36, 154) &= \underline{\qquad} \\ \mathrm{lcm}(154, 476) &= \underline{\qquad} \\ \mathrm{lcm}(442, 476) &= \underline{\qquad} \\ \mathrm{lcm}(154, 476, 4235) &= \underline{\qquad} \end{aligned}$$

# 9 Irrationals That Act Like Rationals

The rational numbers $\mathbb{Q}$ (a.k.a. fractions) are *closed* over addition, subtraction, multiplication, and division (excepting division by zero). The irrationals $\mathbb{R} \setminus \mathbb{Q}$ are *not closed* over the same operations, as seen in the homework.

However, the *quadratic rationals* defined as

$$\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} \,|\, a \in \mathbb{Q}, b \in \mathbb{Q}\}$$

are closed for a given $d$!

Show that the quadratic rationals $\mathbb{Q}(\sqrt{2})$, or numbers of the form $a + b\sqrt{2}$, are closed over addition and division.

- To show these numbers are closed under addition, first try an example. Fill in the blanks below with numbers:

  $$(2 + 10\sqrt{2}) + (3 + 20\sqrt{2}) = \_\_ + \_\_\sqrt{2}.$$

  Now try it symbolically. Fill in the blanks with the correct symbolic expression:

  $$(a + b\sqrt{2}) + (c + d\sqrt{2}) = \_\_\_\_\_\_ + \_\_\_\_\_\_\sqrt{2}.$$

- To show these numbers are closed under division, again start with an example. Compute $x$ and $y$, then fill in the blanks with rational numbers. Hint: $y$ is an integer.

  $$\begin{aligned} \frac{3 + 2\sqrt{2}}{2 + 1\sqrt{2}} &= \frac{3 + 2\sqrt{2}}{2 + 1\sqrt{2}} \cdot \frac{2 - 1\sqrt{2}}{2 - 1\sqrt{2}} \\ &= \frac{x}{y} = \_\_ + \_\_\sqrt{2}. \end{aligned}$$

  Now try it symbolically. Compute the expressions for $x$ and $y$, then fill in the blanks with rational expressions. Again, $y$ has no $\sqrt{2}$ in it.

  $$\begin{aligned} \frac{a + b\sqrt{2}}{c + d\sqrt{2}} &= \frac{a + b\sqrt{2}}{c + d\sqrt{2}} \cdot \frac{c - d\sqrt{2}}{c - d\sqrt{2}} \\ &= \frac{x}{y} = \_\_\_\_\_\_ + \_\_\_\_\_\_\sqrt{2}. \end{aligned}$$

*(Note: Showing that multiplication is closed is about the same as addition above, but it's not included in this question. And closure over division here is an* actual *use for removing irrationals from denominators.)*

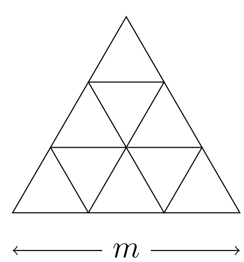
# 10 Translating to Algebra

**Problem 1:** Three people need a motel room for the night. The room is advertised for \$30, so they split it into \$10 apiece. After paying the front-desk lackey, the manager informs the lackey that the price was really \$25. Rather than trying to split \$5 three ways, the lackey takes \$2 for himself and returns \$1 to each person.

Assign variables to the amount each person paid initially, the amount each person paid after the refund, the amount of the refund, and the amount taken by the lackey. Then express the accounting as an algebraic equality.[1]

**Problem 2:** Assume rug remnants can be purchased for \$3.60 per square yard and can be finished at a cost of 12¢ per foot. Express the cost to buy and finish a remnant algebraically in terms of its length and width. Include the units of each constant and each variable. Provide a numerical example to demonstrate that the units work out correctly.

**Problem 3:** Imagine making the triangular pattern below out of toothpicks. How many toothpicks are required to form the pattern with $m$ along the bottom? Can you express the toothpicks required for the $m^{\text{th}}$ case in terms of the $(m-1)^{\text{st}}$ case? *(Hint: Consider stacking triangles.)*




[1]Traditionally, this question is used to confuse people at bars with "where did the extra dollar go?" At the end, it may appear that a dollar is missing because $3 \cdot \$9 + \$2 = \$29$. The algebraic form should show the flaw in that reasoning.

## 11 Slopes: Parallel, Perpendicular, *etc.*

Each row of the following table provides a line in standard form and a point. Given the following lines in standard form and a point for each line, find the equation of a line parallel to the given one through the point and the equation of a line perpendicular to the given line through the point. **All results must be in standard form.**

| Line | Point | Parallel | Perpendicular |
|---|---|---|---|
| $4x - 3y = 12$ | $(2, 3)$ | | |
| $3x + 4y = -12$ | $(-2, -2)$ | | |
| $3x + y = 5$ | $(0, 0)$ | | |
| $x - 3y = -3$ | $(-1, -1)$ | | |

**Describe the pattern you see in the coefficients.**

**Answer the following:**

- What is the slope of a line connecting $(-1, 1)$ and $(1, -1)$?
- What is the slope of a line connecting $(0, 0)$ and $(1, 1)$?
- What is the slope of a horizontal line?
- What is the slope of a vertical line?
- If two lines have different slopes, how often do they intersect?
- If two lines have the same slope, what are the two cases describing their possible intersections?

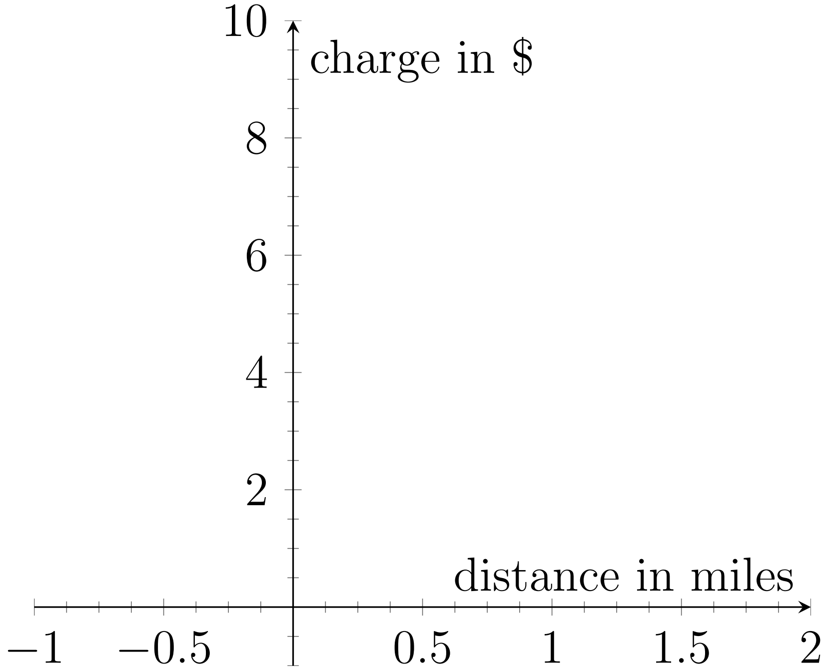
# 12 Taxi Fares: A Linear Model

A summary of taxi fares in four major cities[2]:

| city | initial charge | initial distance | dist. incr. | cost per incr. |
|---|---|---|---|---|
| New York (NY) | 2.50 | 1/5 | 1/5 | 0.40 |
| Miami (MI) | 2.50 | 1/6 | 1/6 | 0.40 |
| Boston (BOS) | 1.75 | 1/8 | 1/8 | 0.30 |
| Cleveland (ICK) | 1.80 | 1/6 | 1/4 | 0.40 |

So for a taxi ride in New York, you pay \$2.50 for the first $1/5^{\text{th}}$ of a mile and \$0.45 per each additional $1/5^{\text{th}}$ of a mile.

**Sketch the lines associated with each taxi fare as a function of distance.** Label each line with the city abbreviation.




**Use your sketch to answer (approximately) the following questions:**

1. Is Miami always more expensive than New York? If not, about where are they equal?
2. Is New York always more expensive than Boston? If not, about where are they equal?
3. Is Boston always more expensive than Cleveland? If not, about where are they equal?

[2]Data from `http://www.schallerconsult.com/taxi/fares1.htm`.