Math 202 Test and Solutions

Algebra and Lines

24 November, 2008

Due in class on 1 December, 2008

Following are eight questions, each worth the same amount. Complete ***six*** of your choice. I will only grade the first six I see. Make sure your name is on the top of each page you return.

Explain your reasoning for each problem whenever appropriate; that helps me give partial credit. Perform scratch work on scratch paper; keep your explanations clean.

Make final answers obvious by boxing or circling them. When a question asks you to construct a table or perform a computation, showing the table or writing out the computation's steps is a part of the question and is **not optional**.

And remember to read and answer the entire question. There is copious explanation before a few problems. The explanation repeats some relevant material from class.

This exam also will be posted at

`http://jriedy.users.sonic.net/VI/math202-f08/`.

Mail me at jason@acm.org with any questions.

Contents

1 Translating to Algebra

Problem 1: Three people need a motel room for the night. The room is advertised for \$30, so they split it into \$10 apiece. After paying the front-desk lackey, the manager informs the lackey that the price was really \$25. Rather than trying to split \$5 three ways, the lackey takes \$2 for himself and returns \$1 to each person.

Assign variables to the amount each person paid initially, the amount each person paid after the refund, the amount of the refund, and the amount taken by the lackey. Then express the accounting as an algebraic equality.[1]

Solution to Problem 1:

Variable	Meaning
P_0	Person's initial payment
P_1	Person's final payment
R	Refund
F	Lackey's take

With the variables to the left,

$$3P_0 - R + F = 3P_1.$$

Here there are three initial contributions, the refund subtracted, then the lackey's take is added to the cost. That is split among the three people paying P_1.

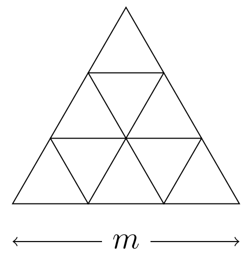
Problem 2: Assume rug remnants can be purchased for \$3.60 per square yard and can be finished at a cost of 12¢ per foot. Express the cost to buy and finish a remnant algebraically in terms of its length and width. Include the units of each constant and each variable. Provide a numerical example to demonstrate that the units work out correctly.

Solution to Problem 2: If the total cost is C dollars, the length is L feet, and the width is W feet, then

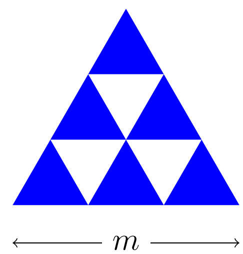
$$C\ \$ = \frac{3.60\ \$}{9\ \text{ft}^2}(L\ \text{ft} \cdot W\ \text{ft}) + \frac{0.12\ \$}{1\ \text{ft}}(2L\ \text{ft} + 2W\ \text{ft}).$$

[1] Traditionally, this question is used to confuse people at bars with "where did the extra dollar go?" At the end, it may appear that a dollar is missing because $3 \cdot \$9 + \$2 = \$29$. The algebraic form should show the flaw in that reasoning.

Problem 3: Imagine making the triangular pattern below out of toothpicks. How many toothpicks are required to form the pattern with m along the bottom? Can you express the toothpicks required for the m^{th} case in terms of the $(m-1)^{\text{st}}$ case? *(Hint: Consider stacking triangles.)*



Solution to Problem 3: When $m = 1$, you need 3 toothpicks for 1 little triangle (in blue, below). For $m = 2$, you need $3 = 2 + 1$ little triangles of 3 toothpicks each, for $9 = 3 \cdot (2 + 1)$ toothpicks. And $m = 3$ needs $6 = 3 + 2 + 1$ little triangles of 3 toothpicks, for $18 = 3 \cdot (3 + 2 + 1)$ toothpicks.



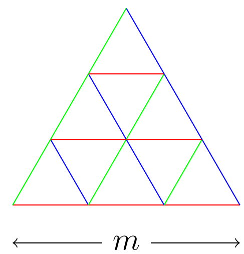
In general, then, the number of toothpicks $T(m)$ is given by

$$T(m) = 3\sum_{i=1}^{m} i = \frac{3m(m+1)}{2}.$$

You also can think of the m^{th} case as stacking m new triangles under the existing triangles. This gives the *recursive* form

$$T(m) = m + T(m-1).$$

Here's another way to reach the same conclusion. Consider the red lines below. For $m = 1$, there is only one red toothpick. For $m = 2$, there are $3 = 1 + 2$ red toothpicks. And $m = 3$ has $6 = 3 + 2 + 1$ red toothpicks. So the number of red toothpicks is $\sum_{i=1}^{m} = m(m+1)/2$. The same applies for the green and blue toothpicks. So the total number of toothpicks again is $T(m) = 3m(m+1)/2$.



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2 Interpolating From a Table

The US Naval Observatory publishes a table of sunrise and sunset times for any location worldwide and any year at `http://aa.usno.navy.mil/data/docs/RS_OneYear.php`. The following data is for 2009 in Bristol, VA. The times ignore daylight savings time.

Date	Sunrise	Sunset
1 January	7:41am	5:24pm
1 February	7:30am	5:55pm
1 March	6:59am	6:23pm
1 April	6:14am	6:51pm
1 May	5:35am	7:17pm
1 June	5:12am	7:42pm

Estimate the following times by interpolating between the closest dates in the above table:

1. Sunrise on 14 February
Solution: With the time in minutes, the nearest points are 1 February's $(32, 450)$ and 1 March's $(60, 419)$. The point we are looking for is at the day number 45. Substituting into the point form,
$$\frac{45-32}{60-32} = \frac{T-450}{419-450}.$$
Solving gives $T \approx 435.61$, or just before **7:16am**.

2. Sunset on 28 January
Solution: With the time in minutes, the nearest points are 1 January's $(1, 1044)$ and 1 February's $(32, 1075)$. The point we are looking for is at the day number 28. Substituting into the point form,
$$\frac{28-1}{32-1} = \frac{T-1044}{1075-1044}.$$
Solving gives $T = 1071$, or at **5:51pm**.

3. Sunset on 28 May
Solution: With the time in minutes, the nearest points are 1 May's $(121, 1157)$ and 1 June's $(153, 1182)$. The point we are looking for is at the day number 148. Substituting into the point form,
$$\frac{148-121}{153-121} = \frac{T-1157}{1182-1157}.$$
Solving gives $T \approx 1159.42$, or just after **7:19pm**.

4. Sunrise on 15 April
Solution: With the time in minutes, the nearest points are 1 April's $(91, 374)$ and 1 May's $(121, 335)$. The point we are looking for is at the day number 105. Substituting into the point form,
$$\frac{105-91}{121-91} = \frac{T-374}{335-374}.$$
Solving gives $T = 355.8$, or just before **5:56am**.

To interpolate, treat the closest two sunrise or sunset times as points (day, time), where the day is the day of the year. So 1 January is day 1, 1 February is day 32, *etc.* Connect the two points by a line and derive a function for the time given the day (also known as the slope-intercept form of the line). Then evaluate that function on the required day.

Is a linear model reasonable? To test this, find the equations of the following lines:

1. connecting 1 January and 1 June,
2. connecting 1 February and 1 April, and
3. connecting 1 March and 1 May.

Are these lines nearly the same? Explain.

Solution: Let's test sunrise times and assume sunset times are similar.

Date range	1 Jan — 1 June	1 Feb — 1 Apr	1 Mar — 1 May
Point form	$\frac{D-1}{152-1}=\frac{T-461}{312-461}$	$\frac{D-32}{91-32}=\frac{T-450}{374-450}$	$\frac{D-60}{121-60}=\frac{T-419}{335-419}$
Slope-intercept	$T=\frac{-149}{151}D+\frac{69462}{151}$	$T=\frac{-76}{59}D+\frac{26518}{59}$	$T=\frac{-84}{61}D+\frac{25499}{61}$
Approx.	$T\approx -0.9868D+460.01$	$T\approx -1.2881D+449.46$	$T\approx -1.3770D+418.02$

The intercepts may be close, but the slopes are massively different. So a linear model likely is not the best.

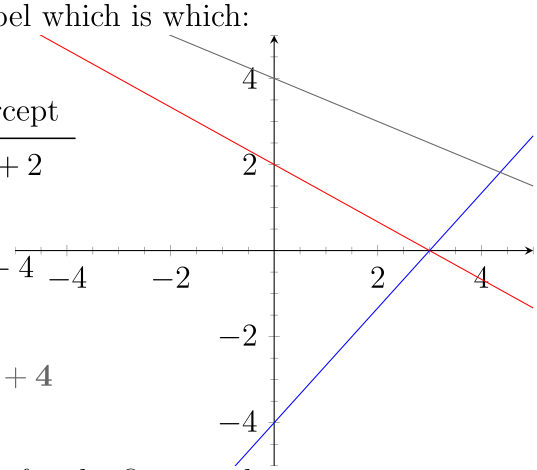
3 Forms of Lines

We discussed many forms of lines in class. Among them are the following, where (x_0, y_0) and (x_1, y_1) are points, m is the slope, x_{int} is the x-intercept, and y_{int} is the y-intercept:

standard form:	$ax + by = c$	slope-intercept form:	$y = mx + y_{\text{int}}$
intercept form:	$\frac{x}{x_{\text{int}}} + \frac{y}{y_{\text{int}}} = 1$	point form:	$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}$

Fill out the rows of the following table by converting the given lines into the other forms, and then **plot the lines** and label which is which:

standard	intercept	slope-intercept
$\mathbf{2x + 3y = 6}$	$\frac{x}{3} + \frac{y}{2} = 1$	$y = \frac{-2}{3}x + 2$
$-4x + 3y = -12$	$\mathbf{\frac{x}{3} + \frac{y}{-4} = 1}$	$y = \frac{4}{3}x - 4$
$\frac{-1}{2}x + y = 4$	$\frac{x}{8} + \frac{y}{4} = 1$	$\mathrm{y} = \frac{-1}{2}\mathrm{x} + 4$



Hint: The intercept form is the easiest to plot for the first two lines.

Write the solution set of each line in a simplified set-builder notation. For example, the solution set of the line $x + y = 1$ is $\{(x, 1 - x) \mid x \in \mathbb{R}\}$ or $\{(1 - y, y) \mid y \in \mathbb{R}\}$. While writing that set as $\{(x, y) \mid x + y = 1, x \in \mathbb{R}\}$ is *technically* correct, I'm looking for one variable as a function of the other.

Line	Solution Set
$2x + 3y = 6$	$\{(\mathbf{x}, \frac{-2}{3}\mathbf{x} + \mathbf{2}) \mid \mathbf{x} \in \mathbb{R}\}$
$\frac{x}{3} + \frac{y}{-4} = 1$	$\{(\mathbf{x}, \frac{4}{3}\mathbf{x} - \mathbf{4}) \mid \mathbf{x} \in \mathbb{R}\}$
$y = \frac{-1}{2}x + 4$	$\{(\mathbf{x}, \frac{-1}{2}\mathbf{x} + \mathbf{4}) \mid \mathbf{x} \in \mathbb{R}\}$

Which form of the line is more useful for the solution set? Answer: **the slope-intercept form**

4 Slopes: Parallel, Perpendicular, *etc.*

Each row of the following table provides a line in standard form and a point. Given the following lines in standard form and a point for each line, find the equation of a line parallel to the given one through the point and the equation of a line perpendicular to the given line through the point. **All results must be in standard form.**

Line	Point	Parallel	Perpendicular
$3x - 4y = 8$	$(8, 11)$	$3x - 4y = -20$	$4x + 3y = 55$
$8x + 7y = -5$	$(-2, 7)$	$8x + 7y = 33$	$7x - 8y = -70$
$\frac{3}{5}x - \frac{6}{5}y = 2$	$(0, 0)$	$\frac{3}{5}x + \frac{6}{5}y = 0$	$\frac{6}{5}x - \frac{3}{5}y = 0$
$\frac{6}{7}x + \frac{1}{7}y = -3$	$(-1, -1)$	$\frac{6}{7}x + \frac{1}{7}y = -1$	$\frac{1}{7}x - \frac{6}{7}y = \frac{5}{7}$

Describe the pattern you see in the coefficients.

Answer: The coefficients for the parallel form are identical, and the coefficients for the perpendicular form are swapped with one coefficient negated.

Answer the following:

- What is the slope of a line connecting $(-1, -1)$ and $(2, 2)$? **The slope is 1.**
- What is the slope of a line connecting $(-1, 1)$ and $(2, -2)$? **The slope is -1.**
- What is the slope of a horizontal line? **The slope is 0.**
- What is the slope of a vertical line? **The slope is undefined.**
- If two lines have different slopes, how often do they intersect? **Exactly once.**
- If two lines have the same slope, what are the two cases describing their possible intersections? **Either never or they are the same line.**

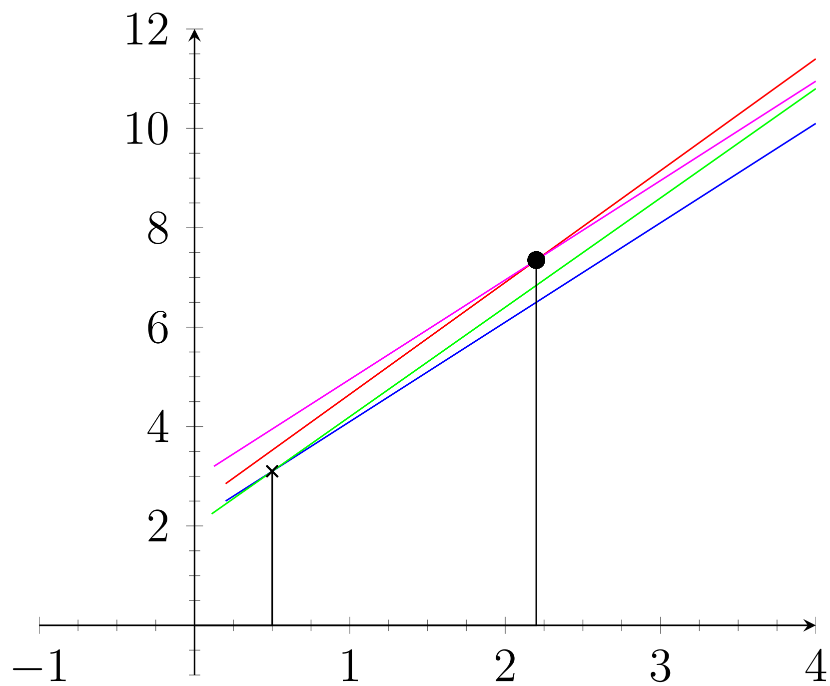
5 Taxi Fares: A Linear Model

A summary of taxi fares in four major cities:

city	initial charge	initial distance	dist. incr.	cost per incr.
San Francisco (SF)	2.85	1/5	1/5	0.45
New York (NY)	2.50	1/5	1/5	0.40
Los Angeles (LA)	2.20	1/11	1/11	0.20
Las Vegas (LV)	3.20	1/8	1/8	0.25

So for a taxi ride in New York, you pay \$2.50 for the first $1/5^{\text{th}}$ of a mile and \$0.45 per each additional $1/5^{\text{th}}$ of a mile.

Sketch the lines associated with each taxi fare as a function of distance. Label each line with the city abbreviation.



Use your sketch to answer (approximately) the following questions:

1. Is Las Vegas always more expensive than San Francisco? If not, about how many miles must you travel for the fares to be equal in San Francisco and Las Vegas? **2.2 miles**

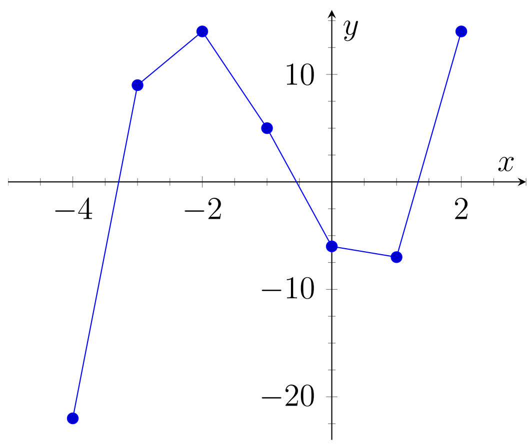
2. About how many miles must you travel for the fares to be equal in New York and Los Angeles? **.5 miles**

3. Are taxis more expensive in San Francisco or Los Angeles? *Hint: See if one line is always above the other.* **San Francisco**

6 Lines and Roots of Polynomials

Consider the polynomial $f(x) = 2x^3 + 5x^2 - 8x - 6$. Fill in the following table by evaluating the polynomial at the provided points. Then plot the line segments between the points $(x, f(x))$.

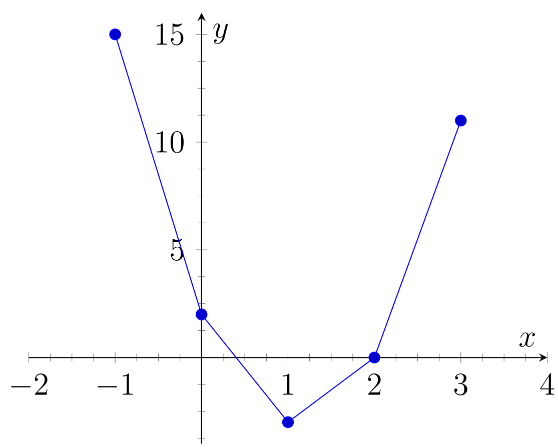
x	$f(x)$
-4	-22
-3	9
-2	14
-1	5
0	-6
1	-7
2	14



In which intervals are there roots of f(x)? In other words, in which intervals does $f(x) = 0$ somewhere within the interval? You do not need to determine the roots here, only the intervals. **Answer: The intervals are $(-1, 0)$ and $(1, 2)$.**

Now consider $g(x) = 4x^2 - 9x + 2$. Evaluate the function at the following points and plot the segments.

x	$g(x)$
-1	15
0	2
1	-3
2	0
3	11



Now find the non-obvious root by bisection. One root, where $g(x) = 0$, will occur at one of the x values in the above table. Take the interval that must contain the other root. Evaluate the function at the half-way point and decide on which side the root must lie. Continue until you find the root where $g(x) = 0$. Verify the roots you find by comparing to those from the quadratic equation,

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Answer: $g(0.5) = -1.5$, $g(0.25) = 0$, so the other root is $\mathbf{1/4}$. This and the obvious root of 2 agree with the quadratic equation.

7 Algebraic Transformations

Justify each line in the following derivation of $1 = 2$, or state the mistake. One of the lines already is justified (and is correct). Also continue the column of numerical examples; that column should help you identify the mistake.

	Reason	Example
$x = y$	Given	$3 = 3$
$x^2 = xy$	2	$3^2 = 3 \cdot 3$
$2x^2 = x^2 + xy$	3	$2 \cdot 3^2 = 3^2 + 3 \cdot 3$
$2x^2 - 2xy = x^2 - xy$	3	$2 \cdot 3^2 - 2 \cdot 3 \cdot 3 = 3^2 - 3 \cdot 3$
$2(x^2 - xy) = 1(x^2 - xy)$	5	$2 \cdot (3^2 - 3 \cdot 3) = 1 \cdot (3^2 - 3 \cdot 3)$
$2 = 1$	6	

1. Substitution using the given information.
2. Multiplication by the same, non-zero quantity.
3. Addition or subtraction of the same quantity.
4. Division by the same, non-zero quantity.
5. Applying arithmetic properties (*e.g.* commutativity or distributivity) on one or both sides.
6. The mistake.

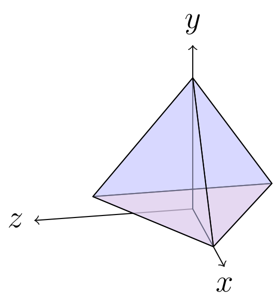
Explain why the mistake is a mistake.

Answer: The quantity $x^2 - xy = 0$, as can be seen through the numerical example $3^2 - 3 \cdot 3 = 0$.

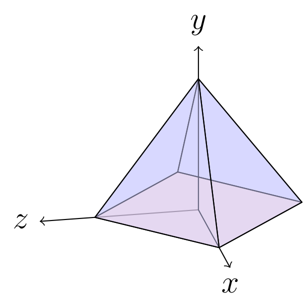
8 Finding an Algebraic Relationship

Fill in the following table for the shapes below. Vertices are the points or corners, faces are the sides, and edges are the lines.

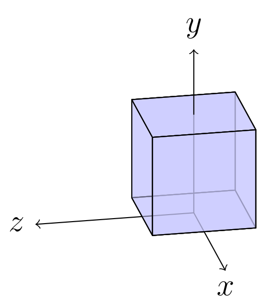
Shape	Vertices	Faces	Edges
Tetrahedron	4	4	6
Pyramid	5	5	8
Cube	8	6	12
Triangular box	6	5	9



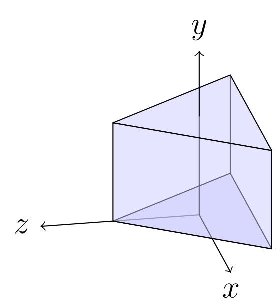
Tetrahedron



Pyramid



Cube



Triangular box

Now find an algebraic relationship between the vertices, faces, and edges. The relationship will be the sum of two of the quantities above equaling the sum of a constant with the remaining quantity. Assign variables to represent each quantity except for the constant, and express the relationship as an algebraic statement.

Answer:

Kind	Variable
Vertices	V
Faces	F
Edges	E

Then $\mathbf{V} + \mathbf{F} = \mathbf{2} + \mathbf{E}$.